Microeconomía II, QED
Teoría de la Decisión, Licenciatura de Matemáticas

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Problem set # 7

Exercise 7.1 Within the Spence signalling model presented in class, construct a *graphical* example where there is *no* pooling Signalling equilibrium (SE).

Exercise 7.2* Recall the labor-market strategic context in four stages described in class.

(a) Model it as a Bayesian game (that is, *not* as a signalling game). Show that, for any education level $\eta_o < \eta^*(H)$, there exists a Bayes-Nash equilibrium (BNE) in which both types of worker choose η_o. (Recall that $\eta^*(\theta)$ stands for the optimal education level chosen by each type $\theta = H, L$ under complete information.)

(b) Recast the strategic context as a signalling game and show that there exists a certain $\check{\eta} > 0$ such that there can be no pooling SE where both types choose an education level $\eta < \check{\eta}$.

(c) Explain the contrast between (a) and (b).

Exercise 7.3 In the Spence signalling model, posit linear production functions of the form:

$$f(\theta, \eta) = a(\theta) + b(\theta)\ \eta$$

with $a(H) \geq a(L) \geq 0$ and $b(H) > b(L)$.

(a) Propose conditions on the cost functions $c(\theta, \cdot)$ that guarantee the existence of a pooling SE for any given value of $p > 0$ (the prior probability of type H).

(b) Show as well that, given any cost functions and any particular η_0, there exists some sufficiently low $\hat{p} > 0$ such that if $p \leq \hat{p}$, there exists *no* pooling SE with equilibrium education level η_0.

Exercise 7.4* In the context of Exercise 6.3, consider any of the pooling equilibria existing under the conditions specified in its Part (a). Let ω_0 and η_0 be the wage and education level materialized at this equilibrium and define the education level $\breve{\eta} \neq \eta_0$ that solves the following equation:

$$\omega_0 - c(H, \eta_0) = [p\ f(H, \breve{\eta}) + (1-p)\ f(L, \breve{\eta})] - c(H, \breve{\eta}),$$

that is, at the education level $\check{\eta}$, the worker of type H is indifferent between being payed the expected productivity or, alternatively, choosing the level η_0 and being payed ω_0.

(a) Prove that $\check{\eta}$ is well-defined and unique if

$$\frac{\partial c\left(H, \eta_0\right)}{\partial \eta} < p\ b(H) + (1-p)\ b\left(L\right)$$

and $\frac{\partial^2 c(H,\eta)}{\partial \eta^2}$ is bounded below, above zero.

(b) Show that the common education level η_0 may be supported as a pooling SE by the following off-equilibrium beliefs (much less extreme than those contemplated in class):

$$\begin{aligned} \mu(\eta)(H) &= p \quad \text{if } \eta \geq \check{\eta}, \\ \mu(\eta)(H) &= 0 \quad \text{if } \eta < \check{\eta},\ \eta \neq \eta_0. \end{aligned}$$

Exercise 7.5 Prove or refute the following assertion:

If the no-envy condition applies in the Spence signalling model, i.e.

$$f\left(L, \eta^*\left(L\right)\right) - c\left(L, \eta^*\left(L\right)\right) \geq f\left(H, \eta^*\left(H\right)\right) - c\left(L, \eta^*\left(H\right)\right). \tag{6.1}$$

the induced signalling game displays a separating SE but has *no* pooling SE.

Exercise 7.6 Under the assumption that (6.1) does *not* hold, find a separating SE for the Spence signalling model that (both on- and off-equilibrium) is different from the one specified in class under these circumstances. Determine the maximum education level for the high type which may be supported at some such equilibrium.

Exercise 7.7* In the context of Exercise 6.3, suppose that $p > 1/2$, make:

$$\begin{aligned} &a\left(\theta\right) = 0, \qquad \theta = H, L \\ &b\left(H\right) = 2, \qquad b(L) = 1. \end{aligned}$$

and posit cost functions given by:

$$c\left(\theta, \eta\right) = \frac{\eta}{b(\theta)}.$$

Characterize *all* SE.

Exercise 7.8 Extend the First-price auction studied in class to a context with three potential buyers and compute the symmetric BNE in affine strategies. Can you extrapolate matters to the general case with n buyers?

Exercise 7.9 Consider a two-buyer allocation setup where buyers' valuations for the single indivisible good are private information and *a priori* chosen from the set $V = \{v^0, v^1\}$ according to probabilities $P(v, v') = 1/4, \forall (v, v') \in V^2$. Define the Bayesian game induced by a First-price auction where bids are restricted to the set $B = \{v^0, v^1, \frac{v^0+v^1}{2}\}$ and find two different BNE.

Exercise 7.10* Consider an allocation setup where n individuals participate in a so-called *Second-Price auction* for a given indivisible object. In this auction, bids are submitted simultaneously and, given any bid profile $(a_1, a_2, ..., a_n) \in \mathbb{R}^n_+$, the object is assigned to the individual (or one of the individuals, if there are several) issuing the highest bid, as in the First-Price auction. However, the individual who obtains the good does *not* pay for it her own bid. Instead, she pays the second highest bid given by $\max\{s_j : s_j \leq s_i\ ,\ j \neq i\}$.

(a) Assuming that each individual is privately informed of her own valuation, model the situation as a Bayesian game and find all of its BNE.
(b) Suppose now that there is a minimum bid $\hat{p} > 0$ that buyers must be prepared to submit if they wish to participate in the mechanism. (If only one buyer chooses to participate, the second-highest bid is identified with $\hat{p}$.) Determine the BNE in this case.

Exercise 7.11* Consider a First-Price auction (as described in Subsection **??**) with three potential buyers and the following additional feature: before the auction starts, the seller can demand from each buyer a deposit $x \geq 0$ to participate in it. However, if a buyer who has committed this deposit does not eventually obtain the good, she is entitled to recover it fully.

In this context, consider two different variations. In the first one, each buyer knows how many others *may* participate in the mechanism (i.e. two more) but not how many are finally involved (that is, who pays the deposit). In the second variation, the information of who is actually involved in the auction is common knowledge before it actually starts.

Which of these two possibilities would the seller choose? Which would be her preferred value of x?