The Evolution of Comparative Advantage*

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July 16, 2003

Abstract

We study the evolution of comparative advantage in a model with two countries and $\ell \geq 2$ goods. We show that the stochastically stable set consists of the pattern of production that follows the chain of comparative advantage in a sufficiently large world economy. Also, the chain of comparative advantage has an elegant structure that allows for step-by-step evolution. So convergence towards the equilibrium where each good is produced in the "right" country is fairly rapid. [*JEL* B52 (evolutionary economics), D50 (general equilibrium), F10 (international trade)]

*The authors would like to thank seminar participants at the European University Institute, Texas A&M University, and the 12th European Workshop on General Equilibrium Theory at Bielefeld University for comments on earlier drafts of this work. Fisher would like to thank the Johns Hopkins Bologna Center and the European University Institute for their hospitality during the time this research was conducted. Vega-Redondo acknowledges support by the Spanish Ministry of Science and Technology, Grant No. BEC2001-0980.

1 Introduction

In a prescient paper, Alchian (1950) argued that the assumptions of profit maximization and perfect foresight were not necessary to describe the long-run behavior of an economic system. He encouraged economists to adopt a biological approach, based upon the principles of evolution and natural selection. Darwinian selection may well insure that firms with the best production techniques survive in the long run. This paper is a formalization of his ideas in a classic model of general equilibrium that has served for almost two centuries as the foundation of international trade theory. In essence, this paper is about Ricardo and Walras meeting Darwin, and we are happy to report that their collective offspring are healthy and fecund.

Assuming that firms are identical *ex ante* and choose quantities in a single market, Vega-Redondo (1997) showed that the Walrasian equilibrium is the unique long-run outcome of a stochastic evolutionary model akin to those of Kandori, Mailath, and Rob (1993) and Young (1993). The intuition for this result is revealing. Assume first that the profile of outputs is such that price is above marginal cost. Then any firm that increases its own output will lower price but increase its market share at the expense of all the other firms. Thus, even though its own profits may well decrease, its relative fitness will rise and the original situation is not evolutionarily stable. Assume second that the profile of outputs is such that price is equal to marginal cost. Then any firm that increases output will cause all firms to have negative profits, but its profits will be the smallest of all. And any firm that decreases output will make all firms have positive profits, but only at the expense of having the smallest market share and relative fitness. Thus the second profile is indeed evolutionarily stable.

This intuition does not carry over to a model of general equilibrium. Now a firm can choose what product to make and how much of it to produce. Vega-Redondo's intuition about quantity choices is still correct if the price-setting mechanism in each market is modeled consistently. But what about the choice of which product to make? If the profile of outputs is such that price is above marginal cost in at least two markets, then a firm that exits one market for another actually raises the profits and thus relative fitness of the firms it leaves behind and also lowers those of the firms it joins. So it may well be that equilibria where prices do not reflect opportunity costs are evolutionarily stable. If this conjecture is true, then Alchian's ideas may not be on solid conceptual foundations. Here are his own words:

> In a static environment, if one improves his position relative to his former position, then the action taken is better than the former one, and presumably one could continue by small increments to advance to a local optimum. An analogy is pertinent. A nearsighted grasshopper on a mound of rocks can crawl to the top of a particular rock. But there is no assurance that he can get to the top of the mound, for he might have to descend for a while or hop to new rocks. (1950, p. 219)

The main result of this paper is that the Alchian's grasshopper does indeed reach the (global) top of the mound. In our model, this summit is the Walrasian equilibrium in which firms produce according to the chain of comparatives advantage. Another important result is that the grasshopper gets to the top of the hill rather quickly for an evolutionary model.

We analyze a model that in essence has defined the field of international trade theory for almost two centuries. Ricardo's labor theory of value has a very convenient supply side: the linear production structure entails that constant opportunity costs define relative prices in any country in which a least two activities are undertaken (namely, in which there is incomplete specialization). This was the structure that McKenzie (1950) first used to prove the existence of a general equilibrium. The essence of the model in a two-country world is that there is a chain of comparative advantage: the goods can be labeled in order of decreasing home-country comparative advantage, and the double factoral terms of trade "break the chain," with the home country producing the whole spectrum of goods in which it has comparative advantage and the foreign country producing all the other ones.[1]

An advantage of our model is that it is a consistent description of an oligopoly in general equilibrium. In particular, there is no numeraire problem[2] since each agent's payoffs are specified as a share of aggregate output. Neary (2003) uses a different approach to integrate oligopoly into general equilibrium in a coherent fashion. He posits that each firm is large in its own industry but negligible in the world economy. Our setup can be interpreted as a game in which firms make strategic choices about which industries to enter. Even though we do not put much emphasis on a strategic interpretation of our model, the long-run states we find can be regarded as suitable approximations of a Nash equilibrium for such a game.

We report two main results. First, production in the stochastically stable set is performed according to the chain of comparative advantage, if the economy is large enough. Let us put the jargon aside for the moment. Our first result means that Alchian's ideas are right in a tightly specified model. We assume that firms are truly myopic and that there are random mutations. A firm is occasionally called upon to revise its choice of activity, and it will imitate the best local one, without any thought for the long-run ramifications of its behavior. Random mutations entail that every once in a while some firm will randomly pick any possible activity in the world economy. Thus without having to assume profit maximization or perfect foresight, we show that the world economy will converge to the unique Walrasian equilibrium no matter how distorted its initial position is. Second, we show that the convergence to the Walrasian equilibrium is rapid. In particular, the chain of comparative advantage has a very elegant structure that allows step-by-step evolution to work in its full glory. Thus even though there may be many potential activities for every firm in the world economy, the system will not have to wait until "all the ducks are lined up right"

[1] Thus there can be at most one good that is produced both at home and abroad.
[2] The classic reference is Gabszewicz and Vial (1972).

to produce the correct array of goods at home and abroad. Instead, we will see the chain of comparative advantage unwinds link by link, with each step in this process reinforcing the last.

The rest of this paper is structured as follows. In the next section we describe the model and its dynamics. In the third section, we describe the exact relationship between comparative advantage and the long-run outcome of the evolutionary process that our model describes. In the fourth section, we give a discussion of the waiting times for the long-run outcome to arise, and we work through a series of revealing examples and simulations. The fifth section presents our conclusions, describing some of the models strengths and weaknesses and making suggestions for future research.

2 The model

The presentation of the model is divided in two parts. First, we introduce the economic environment. Second, we describe the evolutionary dynamics.

2.1 The economy

The economy is modelled as in the classical Ricardian theory of international trade. There are two countries. An agent anywhere in the world is a consumer-producer endowed with one unit of labor. She can apply this unit of labor (assumed indivisible for simplicity) to one of ℓ different productive activities. Each of the ℓ goods is produced according to a Ricardian technology that differs between countries. Thus the technology of the home country is summarize by a vector of labor coefficients $a^H \in \mathbb{R}^{\ell}_{++}$ with the interpretation that it takes a^H_i hours in the domestic economy to produce one unit of good i. The analogous labor coefficients for the foreign country are $a^F \in \mathbb{R}^{\ell}_{++}$, and we adopt the usual convention that the activities are labeled so that

$$a^H_1/a^F_1 < ... < a^H_\ell/a^F_\ell. \tag{1}$$

Expression (1) lists the goods in order of strictly decreasing domestic comparative advantage.

Let $X = \mathbb{R}^{\ell}_{+}$ be the consumption set for each agent. Agents anywhere in the world have identical homothetic and additively separable preferences summarized by a utility function

$$u : X \to \mathbb{R}$$

that is increasing, strictly quasi-concave, and once continuously differentiable on the interior of X. Since this function is additively separable, the gradient of u can be written as $\nabla u(x) = (\frac{\partial u}{\partial x_1}(x_1), ..., \frac{\partial u}{\partial x_\ell}(x_\ell))$ with each $\frac{\partial u}{\partial x_i}(\cdot)$ continuously differentiable at every $x_i > 0$. This assumption on preferences is general enough to incorporate the class of constant-elasticity-of-substitution utility functions, and it is slightly less general than the norm of identical homothetic preferences in international trade theory. It includes Mill-Graham preferences, the historical norm in the literature on international trade in a Ricardian framework.

These assumptions about preferences also ensure the stability of the dynamics of quantity adjustment in a neighborhood of the Walrasian equilibrium.

There are N^H domestic agents and N^F foreign ones. Let

$$z^H(t) = (z_1^H(t), ..., z_\ell^H(t))'$$

where $z_i^H(t)$ is the integral number of domestic producers in sector i at time t. Also, let $z^F(t)$ denotes its counterpart for the foreign country. The state of the world economy at time t is a vector

$$z(t) = (z^H(t)', z^F(t)')'$$

summarizing the number of domestic and foreign agents in each of the productive activities. The state space is

$$Z = \{z \in \mathbb{Z}_+^{2\ell} : \sum_{i=1}^{\ell} z_i^H = N^H \text{ and } \sum_{i=1}^{\ell} z_i^F = N^F\}. \tag{2}$$

Each state induces a vector of aggregate supplies for the world economy. The vector of outputs for the home economy at any t is just

$$y^H(t) = (z_1^H(t)/a_1, ..., z_\ell^H(t)/a_\ell)',$$

and the analog for the foreign economy is $y^F(t)$. The world aggregate supplies are given by $y(t) = y^H(t) + y^F(t) \in \mathbb{R}_+^\ell$. The state space described in (2) has a natural interpretation as a discrete distribution of domestic and foreign agents (or firms) across the several activities in their own countries.

Let $I(t) \subset \{1, 2, ..., \ell\}$ be the set of goods in strictly positive supply at t, i.e. $i \in I(t)$ if $y_i(t) > 0$. For good $j \notin I(t)$ not produced at t, it is convenient to set $p_j(y_j(t)) = 0$. Thus we consider the simple rule

$$p_i(z(t)) = \begin{cases} \frac{\partial u}{\partial x_i}(y_i(t)) & if \quad i \in I(t) \\ 0 & otherwise \end{cases} .$$

and posit that the list of prices is simply the vector

$$p(z(t)) = (p_1(z(t)), ..., p_\ell(z(t)))'$$

or any other vector proportional to it. Let us write:

$$\sigma_i^H(t) = \frac{p_i(z(t))/a_i^H}{p(z(t)) \cdot y(t)}.$$

This expression is well defined because $p(z(t)) \cdot y(t) > 0$. It is the share of world output that accrues to someone who produces $1/a_i^H$ units of the *i-th* commodity. Then the payoff to an agent producing good i in the home economy is:

$$v_i^H(z(t)) = u(\sigma_i^H(t)y(t)).$$

The analog for the payoff to a foreigner producing good i is immediate:

$$v_i^F(z(t)) = u(\sigma_i^F(t)y(t)).$$

where

$$\sigma_i^F(t) = \frac{p_i(z(t))/a_i^F}{p(z(t)) \cdot y(t)}.$$

The main advantage of the above formulation is that it avoids the numeraire problem that has plagued general equilibrium theory under oligopoly.

At first glance, it may seem strange to set the price of goods not produced to zero. But it should be clear that we are simply normalizing the fitness of an activity that is not yet extant at zero. In this light, a natural interpretation of the introduction of new goods would imply that the first entrant into a new activity has a high fitness (and therefore remains active) only to the extent that the new good commands a high price. On the other hand, the fitness of a hypothetical activity not currently in practice is of no evolutionary relevance whatsoever and can thus be simply set to zero.

It is easy to see that the relative fitness of domestic agents producing goods i and j satisfies

$$v_i^H(z(t)) \geq v_j^H(z(t)) \Leftrightarrow \sigma_i^H(t) \geq \sigma_j^H(t) \Leftrightarrow p_i(z(t))/a_i^H \geq p_j(z(t))/a_j^H \tag{3}$$

Since $u(.)$ is weakly monotone, the above expression merely states that an agent in the domestic economy has *higher fitness* in activity i relative to j if and only if the market value of the former is higher than that of the latter. Thus, activity i has an equal fitness with activity j for a domestic agent if, and only if, $p_i(z(t))/a_i^H = p_j(z(t))/a_j^H$, which is precisely the condition for incomplete specialization in the classical Ricardian model.

2.2 Evolutionary dynamics

Time is discrete and indexed by $t \in \{0, 1, 2, \ldots\}$. The evolutionary dynamics is standard, consisting of imitation and mutation. At every t, *one* individual in the world economy is chosen at random with probability $1/N = 1/(N^H + N^F)$. This agent is given a chance to revise her production choice, which she can do either by imitation or mutation.

With probability $1 - \varepsilon > 0$, the agent imitates the fittest local activity. In that case, her choices are restricted to be among those which were active in her own country during the preceding period $t - 1$. Let $I^H(t-1)$ and $I^F(t-1)$ denote the set of those activities in the home and foreign country respectively. Thus, if the agent belongs to the home country, her new chosen activity must lie in the set

$$B^H(t) = \arg \max_{i \in I^H(t-1)} \{v_i^H(z(t-1))\},$$

while if she is foreign it must be in

$$B^F(t) = \arg \max_{i \in I^F(t-1)} \{v_i^F(z(t-1))\}.$$

If several activities give rise to highest fitness, we just assume that any of these is chosen with equal probability. For simplicity, we also postulate that an agent whose activity yields highest fitness does not change to any other activity, even if others exist with the same maximum fitness.

Imitation, therefore, is assumed to operate within the boundaries of each country, in the spirit of the evolutionary model with local interaction proposed by Eshel, Samuelson, and Shaked (1998). This aspect of the model is motivated by the basic premise of the Ricardian paradigm, which posits that international trade reflects technological differences between countries. These differences translate into a country-specific scope for imitation. If the technology (or the underlying implicit resource it uses) is not common to both countries, the returns of the different activities must be assessed separately. This implies, in particular, that an agent who is striving to imitate the most rewarding activity should only be responsive to payoff information in her own country.

The mutation component of the evolutionary dynamics is also quite standard. At each t, the randomly chosen agent is subject to a perturbation (i.e., she "mutates" or "experiments") with probability $\varepsilon > 0$. If this event occurs, she chooses any of the ℓ possible activities with uniform probability. In particular, she can try an activity that has not been used recently in her local economy.

Our assumption that only one agent in the world can change her output decisions in each period entails that economic time flows quickly. In particular, we are ruling out that any industry with several active agents might shut down suddenly if all the agents revise their product choices at the same time. We think that this specification of economic time is natural because, for example, it is quite unlikely that many firms would simultaneously exit a senescent industry without any awareness of how this move impinges on prices. Heuristically, our approach is akin to positing a continuous-time framework without its technical complications.

3 Stochastic stability and comparative advantage

The evolutionary process described above defines a Markov chain (Z, Q^ε), where Z is the state space and Q^ε is the transition matrix parametrized by the value of the mutation probability ε. The canonical element of the Markov matrix $q^\varepsilon_{zz'}$ gives the probability that the unperturbed process that is currently in state z will be in state z' in the next period. If $\varepsilon > 0$, the evolutionary process is ergodic and thus displays a unique invariant distribution μ^ε. As customary, our focus is on μ^ε for small $\varepsilon > 0$. Or, more precisely, we want to characterize the *stochastically stable states* Ω^*, defined as the set of all those states that are in the support of the invariant distribution $\mu^* \equiv \lim_{\varepsilon \to 0} \mu^\varepsilon$. That is:

$$\Omega^* \equiv \{z \in Z | \mu^*(z) > 0\},$$

where the interpretation is that only these states will be observed a significant fraction of time in the long run when the mutation probability ε is small.

The unperturbed dynamic (involving imitation alone when $\varepsilon = 0$) defines a Markov process (Z, Q^0). Let Ω^0 be the union of all limit sets (or recurrent classes) of (Z, Q^0). By the upper hemi-continuity of μ^ε with respect to ε, it is easy to see that $\Omega^* \subset \Omega^0$. (See Kandori, Mailath, and Rob (1993) and Young (1993)). Thus, a natural first step in the characterization of Ω^* is to understand what states in Z are possible candidates for stochastic stability.

Our first observation is that many patterns of complete specialization define absorbing states (and thus limit sets) of the unperturbed dynamics.

Lemma 1 *Let $I^H \subset \{1, ..., \ell\}$ and $I^F \subset \{1, ..., \ell\}$ be any two non-empty sets that partition the index set of sectors. Then there is a limit set $L \subset \Omega^0$ of the imitation dynamics where for each $z(t) = (z^H(t)', z^F(t)')' \in L$,*

$$\begin{aligned} z_i^H(t) &> 0 \Rightarrow i \in I^H \\ z_i^F(t) &> 0 \Rightarrow i \in I^F. \end{aligned}$$

This result is a direct consequence of the fact that imitation alone can not populate a sector from scratch in either of the two countries. This lemma has the unfortunate consequence that rather "perverse" patterns of production could persist for ever in the absence of mutation or experimentation.

In fact, the conclusion can be even more disturbing: inefficient specialization may arise and consolidate with positive probability from interior states where *a priori* all possible patterns can be reached by the imitation dynamic. To illustrate how this may happen, consider a state z where $z_i^v > 0$ and $z_j^v > 0$ for each $v \in \{H, F\}$ and some i and j with $i < j$. According to our convention, the home country has a comparative advantage in i so that the foreign economy should have positive output in this sector only if the home economy does. Suppose, however, that z_i^H is small (say $z_i^H = 1$) while $z_i = z_i^H + z_i^F$ is large relative to z_j. Then, we may well have that $p_i(z)/p_j(z) < a_i^H/a_j^H$, which implies that with probability no lower than $1/N$ the subsequent state z' will have no firms from the home country in sector i. This, of course, implies that the efficient pattern of specialization will never be attained through imitation alone, even though it was possible *a priori*.

The efficient pattern of production following the chain of comparative advantage occurs in a Walrasian equilibrium. Since labor is indivisible in our model, such an equilibrium may well fail to exist. Still, it seems natural to focus on the Walrasian equilibrium of the analogous economy without indivisibilities as a benchmark for our own model.. One may reasonably conjecture that, if the number of agents is large, such a benchmark may act as the "anchor" of a relatively small set of allocations in the economy with indivisibility. This set could then play a role analogous to the Walrasian equilibrium in the setup with divisible labor.

To pursue this approach, let $(p^*, y^{*H}, y^{*F}) \in \mathbb{R}_+^{3\ell}$ be the Walrasian equilibrium prices and outputs in the *benchmark* economy where firms' production choices are *continuous*. Under our maintained assumptions, it is well known that this Walrasian equilibrium exists and is unique. Equivalently, we may think of the benchmark economy as consisting of a continuum of agents, each of whom is infinitesimal and is endowed with a unit of labor. Then the vector of labor allocations induced by the Walrasian equilibrium

$$z^* = (a_1^H y_1^{*H}, ..., a_\ell^H y_\ell^{*H}; a_1^F y_1^{*F}, ..., a_\ell^F y_\ell^{*F})'$$

can be reinterpreted as the measure of agents distributed across the different sectors in the economy.

In a Walrasian equilibrium, there is a sector $\tilde{k}$ such that the home country produces none of the goods outside

$$\tilde{I}^H = \{i \in \mathbb{N} : 1 \leq i \leq \tilde{k}\} \tag{4}$$

while the foreign country produces none outside the set

$$\tilde{I}^F = \{i \in \mathbb{N} : \tilde{k} \leq i \leq \ell\}. \tag{5}$$

Thus there is at most one good that is produced in both countries. It also true that

$$p_i / a_i^H = p_j / a_j^H$$

for goods i and j both produced at home, while the analog is true for any pair of goods produced abroad.[3]

Let $\|\cdot\|$ denote the supremum norm on $\mathbb{R}^{2\ell}$ and let $B_r(z^*) = \{z \in \mathbb{R}_+^{2\ell} : \|z - z^*\| \leq r\}$ be a corresponding ball of radius r in that norm that is centered on the benchmark Walrasian vector z^*. Let $\tilde{Z}^H = \{z \in Z : i \notin \tilde{I}^H \Rightarrow z_i^H = 0\}$ and $\tilde{Z}^F = \{z \in Z : i \notin \tilde{I}^F \Rightarrow z_i^F = 0\}$. These are subsets of the state space in which the pattern of production follows the chain of comparative advantage. Write $\tilde{Z} = \tilde{Z}^H \cap \tilde{Z}^F$ and let

$$M_1(z^*) = \tilde{Z} \cap B_1(z^*)$$

be the smallest subset of the state space that is sure to "bracket" the allocation of firms that prevail in the Walrasian equilibrium of the continuum benchmark economy. The next lemma shows that, if our process starts in $M_1(z^*)$, the imitation dynamics alone never leads it to abandon some $M_q(z^*) = \tilde{Z} \cap B_q(z^*)$ for a fixed $q > 0$ that is independent of the number of firms in either country.

Lemma 2 *Assume that preferences are represented by a utility function that is homothetic, smooth, increasing, strictly quasi-concave, and additively separable. Then, if the world economy is large enough, there is some integer $q > 0$ independent of N^H and N^F such that the neighborhood $M_q(z^*)$ contains a limit set for the imitation dynamics.*

[3] The classic descriptions of this kind of economy are Dornbusch, Fischer, and Samuelson (1977), Jones (1961), and Wilson (1980).

Proof. We will show that this lemma is true as long as $N^H \geq \bar{N}^H$ and $N^F \geq \bar{N}^F$, for suitably large $\bar{N}^H$ and $\bar{N}^F$.

The assumptions on preferences imply that the following condition is true for any good i at $z \in \tilde{Z}$:

$$\frac{z_i^H/a_i^H + z_i^F/a_i^F}{z_i^{*H}/a_i^H + z_i^{*F}/a_i^F} = y_i(z)/y_i^* < 1 \Leftrightarrow p_i(z)/p_i^* > 1. \tag{6}$$

Obviously, the following two conditions also hold for any $z \in \tilde{Z}$:

$$v_i^H(z)/v_i^{H*} > 1 \Leftrightarrow p_i(z)/p_i^* > 1 \qquad (i \in \tilde{I}^H) \tag{7}$$

$$v_i^F(z)/v_i^{F*} > 1 \Leftrightarrow p_i(z)/p_i^* > 1 \qquad (i \in \tilde{I}^F) \tag{8}$$

Note that

$$\sum_{i=1}^{\tilde{k}} z_i^H(t) = \sum_{i=1}^{\tilde{k}} a_i^H y_i^{*H} = N^H \tag{9}$$

$$\sum_{i=\tilde{k}}^{\ell} z_i^F(t) = \sum_{i=\tilde{k}}^{\ell} a_i^F y_i^{*F} = N^F, \tag{10}$$

both of which are integers. These full employment conditions imply that if some sector i in country v has a number of firms above the Walrasian level, then some other sector i' in country v' (where, of course, i may be equal to i' or v equal to v') must have the number of firms below its Walrasian level.

Let $t = 0$, and consider any path $\{z(t)\}_{t=0}^{\infty}$ with $z(0) \in M_1(z^*)$. We now rely on (6) to show that there is some integer $q > 0$ such that $z(t) \in M_q(z^*)$ for all $t > 0$ along any path of the unperturbed process.

Consider first the simpler case where $\tilde{I}^H \cap \tilde{I}^F = \varnothing$ so that there is complete specialization in the Walrasian equilibrium. In that case, note that if an activity $i \in \tilde{I}^v$ has $z_i^v(t) > z_i^{*v}(t)$ at any t, then some other $j \in \tilde{I}^v$ (for the same country v) must have $z_j^v(t) < z_j^{*v}(t)$. This implies that no firm in country v can switch to sector i at t. Since $z_i^v(0) - z_i^{v*}(0) \leq 1$,

$$z_i^v(t) - z_i^{*v}(t) \leq 1 \tag{11}$$

for each $v \in \{H, F\}$, every $i \in \tilde{I}^v$ and all $t \geq 0$. Thus the full employment conditions (9) and (10) imply that

$$-\ell + 1 \leq z_i^v(t) - z_i^{v*}(t) \tag{12}$$

for each $v \in \{F, H\}$, every $i \in \tilde{I}^v$, and all $t \geq 0$. Then (11) and (12) imply the desired conclusion for $q = \ell - 1$.

Consider next the case of incomplete specialization where $\tilde{I}^H \cap \tilde{I}^F \neq \varnothing$, and let $\tilde{k}$ be the sector that is active in both countries. The problem now is that even if $z(0) \in M_1(z^*)$ the firms in one of the countries can "leak" into the marginal sector $\tilde{k}$, forcing all the activities in the trading partner v to have too many firms (or vice versa). But this requires that $z_i^v(t) > z_i^{*v}(t)$ for all other $i \in \tilde{I}^v \backslash \{\tilde{k}\}$ and, for the common sector $\tilde{k}$,

$$z_{\tilde{k}}^H(t) + z_{\tilde{k}}^F(t) > z_{\tilde{k}}^{*H} + z_{\tilde{k}}^{*F}. \tag{13}$$

Let $\xi_i^v(t) \equiv z_i^v(t) - z_i^{*v}$ be the difference in sector i of country v between the number of firms at time t and the Walrasian level, and denote $[\xi_i^v(t)]_+ \equiv \max\{0, \xi_i^v(t)\}$.

The argument for the first case implied that if $\sum_{i \in \tilde{I}^v} [\xi_i^v(t)]_+$ grows over time, then $\sum_{i \in \tilde{I}^v \backslash \{\tilde{k}\}} \xi_i^v(t)$ must also grow. In fact, it is clear that after a certain period we must have

$$\sum_{i \in \tilde{I}^v} \left[\xi_j^v(t)\right]_+ = \sum_{i \in \tilde{I}^v \backslash \{\tilde{k}\}} \xi_i^v(t) = -\xi_{\tilde{k}}^v(t).$$

Since (13) must hold along such a path, it follows that $\xi_{\tilde{k}}^{v'}(t)$ has to rise in country $v' \neq v$. But then the full employment conditions (9) and (10) imply that $\xi_j^{v'}(t)$ for some sector $j \in \tilde{I}^{v'}$ would have to fall to an arbitrarily low negative level. This in turn would stop the flow into sector $\tilde{k}$ of the firms in country v'. In fact, because the utility function is homothetic and additive separable, an upper bound on $\xi_{\tilde{k}}^{v'}(t)$ can be set that depends only on the output ratios y_i^*/y_j^*, $i, j \in \{1, 2, ..., \ell\}$, the outputs in the Walrasian equilibrium. Since these ratios are maintained if the economy is replicated by increasing the number of firms in each country proportionally, it follows that, for sufficiently large N^H and N^F, there is an integer q, independent of $N = N^H + N^F$such that

$$\sum_{j \in \tilde{I}^v} \left[\xi_j^v(t)\right]_+ \leq q$$

for each $v \in \{H, F\}$ and all $t \geq 0$, provided that $z(0) \in M_1(z^*)$. In view of the full employment conditions (9) and (10), this implies that $z(t) \in M_q(z^*)$. ■

Lemma 2 establishes that there is a set of states "around" the Walrasian benchmark profile z^* which is absorbing under the operation of the imitation dynamics. More specifically, if we denote by $L_1, ..., L_n$ the collection of limit sets of the imitation dynamics that satisfy $L_i \subset M_q(z^*)$ for each $i = 1, 2, ..., n$, then the key implication is that the union of these "Walrasian limit sets"

$$\tilde{\Omega} \equiv \cup_{i=1}^n L_i$$

is non-empty. Sometimes, we may find that $\tilde{\Omega}$ is a singleton, as it occurs for example in the simple case with just two goods when the Walrasian equilibrium induces complete specialization according to comparative advantage.

Next, we establish an important complement to Lemma 2. We show that as long as the process starts in the interior of $\tilde{Z}$ (the set where production follows the chain of comparative advantage), there is positive probability that the imitation dynamics enter the set $M_1(z^*)$. Hence, by virtue of Lemma 2, there is positive probability of becoming absorbed by $M_q(z^*)$.

Lemma 3 *Let preferences satisfy our maintained assumptions and suppose that the world economy is large enough. Then, if the process starts at some* $z(0)$ *in the interior*[4] *of* $\tilde{Z}$*, there is positive probability that* $z(t) \in M_1(z^*)$ *at some future* $T \geq 0$.

Proof. Take any $z(0) \in \tilde{Z}$ and consider an ensuing path where, at each t, a firm is selected for adjustment among those whose fitness is not currently a local maximum. More specifically, let us always choose a firm in sector i in country v where the difference $z_i^v(t) - z_i^{*v}$ is maximum. This selection has positive probability and therefore the resulting path has positive probability as well. Moreover, it always involves a sector i and country v where $z_i^v(t) - z_i^{*v} \geq 0$. Thus, there must be a finite T where $z(T) \in M_1(z^*)$. ■

Now we turn to our main result. We establish that when the mutation rate ε is small the process spends most of the time in the neighborhood $M_q(z^*)$. We show, therefore, that the stochastically stable states satisfy $\Omega^* \subset \tilde{\Omega}$. But first we need to introduce some useful concepts and techniques described by Ellison (2000); Vega-Redondo (2003) gives a simplified account of these ideas.

Define a cost function $c : Z \times Z \to \mathbb{N} \cup \{0, \infty\}$ on the set of possible transitions. For every pair $z, z' \in Z$, $c(z, z')$ indicates the *minimum* number of mutations needed to connect z to z' by a (finite) path. Thus, $c(z, z') = 0$ indicates that there is a transition from z to z' through imitation alone, while $c(z, z') = \infty$ means that there is no path at all (even one allowing for mutation) that connects z to z'.

Once the cost function has been specified, the useful notions of radius and coradius of a limit set can be defined. Recall that Ω^0 denotes the the union of all limit sets for the unperturbed dynamics. For any non-empty subset $Y \subset \Omega^0$, define its *basin of attraction* by $A(Y) \equiv \{z \in Z : c(z, y) = 0 \text{ for some } y \in Y\}$. This is the set of states from which the unperturbed (imitation) dynamics leads to some state in Y with positive probability.

Definition 1 *Let* $U \subset Z$ *be a union of limit sets of the imitation dynamic. The* radius *of* U *is*

$$R(U) \equiv \min_{(z,z') \in U \times A(Z \setminus U)} c(z, z').$$

Definition 2 *Let* $U \subset Z$ *be a union of limit sets of the imitation dynamic. The* coradius *of* U *is*

$$CR(U) = \max_{z \notin U} \min_{z' \in U} c(z, z').$$

The radius of a set U is a good measure of how difficult or unlikely it is to exit from that limit set by mutation. Conversely, its coradius measures how difficult it is to enter the set U by focusing on a worst-case scenario. Ellison (2000) combines both notions to establish the following useful result, as applied to the present context

[4] To be precise, this is the relative interior of $\tilde{Z} \subset \mathbb{Z}^{2\ell}$, the set $\{z \in Z : z_i^v > 0 \Leftrightarrow i \in \tilde{I}^v,\ v = H, F\}$.

Theorem 1 (Ellison, 2000) *Let $U \subset Z$ be a union of limit sets of the imitation dynamic such that $R(U) > CR(U)$. Then, the set of stochastically stable states $\Omega^* \subset U$.*

In fact, the above conclusion can be strengthened significantly if one relies on what Ellison calls the *modified coradius.* Assume that a path from a certain z in some limit set L_0 passes through several other limit sets $L_1, ..., L_n$ before reaching a final state z' in some limit set L_{n+1}. (In principle, a limit set can appear on the list several times but not successively.) Then the *modified* cost function $\hat{c}(\cdot, \cdot)$ is computed as follows:

$$\hat{c}(z, z') = c(z, z') - \sum_{i=1}^{n} R(L_i). \tag{14}$$

Hence, we have the following respective counterparts of Definition 2 and Theorem 1 .

Definition 3 *Let $U \subset Z$ be a union of limit sets of the unperturbed (imitation) dynamics. The* modified coradius *of U is*

$$\widehat{CR}(U) = \max_{z \notin U} \min_{z' \in U} \hat{c}(z, z').$$

Theorem 2 (Ellison, 2000) *Let $U \subset Z$ be a union of limit sets of the imitation dynamics such that $R(U) > \widehat{CR}(U)$. Then the set of stochastically stable states $\Omega^* \subset U$.*

The above result is important because it shows the advantages of step-by-step evolution. Think of how difficult it is for a structure as complicated as the human eye to evolve from a cluster of nerve endings in an ancestral nematode. Such an evolutionary process does not appear so improbable if each mutation along the path makes the organism that possesses it genetically fitter. Imagine that it takes one million different mutations for the human eye to evolve, and assume that each mutation occurs with independent probability $\varepsilon > 0$. Then the probability the human eye will evolve is not of the order $\epsilon^{1000000}$ but instead of order ε, as long as each step is reinforcing. The beauty of the chain of comparative advantage is that it fits in very well with an analogous theory of step-by-step evolution. In particular, even if the world economy starts in a very skewed distribution of firms among the productive activities, each step closer to the chain of comparative advantage is actually reinforcing.

An integral part of our argument is to show that the modified coradius of the Walrasian limit set is 1. We will delay the statement and proof of this fact until we have established four small steps that lead up to it. Its essential idea is that from any limit set U_0 it is possible to construct a chain of limit sets $U_1, U_2, ..., U_r$ with the following two features. First, for each $i = 1, ..., r$, it is possible to transit from U_{i-1} to U_i by relying on a single suitably selected mutation and the ensuing operation of the imitation dynamics alone. Second, the last limit set in the chain satisfies $U_r \subset \tilde{\Omega}$.

Throughout the rest of this section, we shall rely on the assumption that the world economy is large enough to abstract from the effects that a single firm can have on the terms of trade. This assumption means that having only one firm switch activities will not cause any relative price to move so much that it jumps across one of the strict inequalities in the chain of comparative advantage (1). Because we have made no restrictions on absolute advantage (the level of a_i^v), having one firm switch can actually cause a large supply shift. Thus we are assuming that the number of firms in the world economy $N = N^H + N^F$ is large enough to abstract from these swings. It should be readily apparent how the arguments would be adapted if arbitrarily small terms-of-trade effects were taken into account.

Let U be a limit set of the imitation dynamics. We denote by $I^H(U)$ and $I^F(U)$ the set of goods that are produced in some state $z \in U$ in the home and foreign country, respectively, and let $I(U) \equiv I^H(U) \cup I^F(U)$. As mentioned previously, we now establish four small steps.

Lemma 4 *Let U be a limit set of the imitation dynamic such that there is some good $i \notin I(U)$. Then a single mutation is enough to have the unperturbed process enter with positive probability into a limit set U' with $i \in I(U') \supset I(U)$.*

Proof. Equation (6) implies that every good that is produced in relatively small amount is bound to increase its output and remain in production in any ensuing limit set. ■

Lemma 4 just states that the introduction of a new good has to give high fitness for the first producer who comes up with that activity. Thus "innovators" will flourish and soon enough every good will be produced somewhere in the world economy.

Lemma 5 *Let U be any limit set of the imitation dynamics. Then, $I^H(U) \cap I^F(U)$ is either empty or consists of at most one sector.*

Proof. Suppose that two different goods i and j are produced in a limit set of the imitation dynamic in both countries. Then in both countries the fitness entailed by producing good i or j must be (almost) the same for any state of the limit set (if the world is large). But this is incompatible with the fact that, by the chain of comparative advantage (1) implies

$$a_i^v / a_j^v \neq a_i^{v'} / a_j^{v'}$$

for $v, v' \in \{H, F\}$, $v \neq v'$. ■

Lemma 5 states simply that there can be at most one common good that is produced in both countries. This is a deep property of Ricardian models of production, and it applies to this and many other models of this genre. In fact the relative fitness of domestic to foreign firms in the production of this common good plays the same role as relative wages do in the traditional model.

Lemma 6 *Let U be a limit set of the imitation dynamics, and consider two goods $i < j$. Assume that $i \in I^F(U) \backslash I^H(U)$ and $j \in I^H(U) \backslash I^F(U)$. Then a single mutation suffices to have the unperturbed process enter with positive probability into a limit set U' where either $i \in I^H(U')$ or $j \in I^F(U')$.*

Proof. Consider any state $z \in U$ and let $p(z)$ be the induced price vector. There are three possibilities:

$$\frac{p_i(z)}{p_j(z)} \leq \frac{a_i^H}{a_j^H} < \frac{a_i^F}{a_j^F} \tag{15}$$

$$\frac{a_i^H}{a_j^H} < \frac{p_i(z)}{p_j(z)} < \frac{a_i^F}{a_j^F} \tag{16}$$

$$\frac{a_i^H}{a_j^H} < \frac{a_i^F}{a_j^F} \leq \frac{p_i(z)}{p_j(z)} \tag{17}$$

If (15) applies, a mutation where a single foreign firm switches to producing good j leads to the desired path with positive probability. This happens because the mutant firm obtains a fitness that is approximately equal to $p_j(z)/a_j^F$, which is higher than that in any other sector in the foreign country (approximately $p_i(z)/a_i^F$). Then a line of argument parallel to that of Lemmata 2 and 3, where we now restrict our attention to set of currently active sectors in the world economy, implies the stated conclusion.

Assume next that (16) is true. Then an analogous argument shows that a transition to a limit set with the desired characteristics can be triggered *either* by a mutation such that a domestic country begins to produce good i *or* by one where a foreign firm starts producing good j.

Finally, the case for (17) is fully symmetric to that for (15). Now a single mutation by a domestic firm into sector i suffices. ■

The proof of Lemma 6 uses the following important fact. Since preferences can be represented by an additively separable utility function, the analysis may concentrate on the goods that are actually currently produced and restrict to a "Walrasian equilibrium" of the world economy defined in terms of these goods. Even if only a few (but not all) sectors are active in the world economy, preferences and production sets are well specified on the reduced commodity space. Such a "restricted" world economy also has a unique Walrasian economy. Thus the general model has an attractive structure that allows the commodity space to become richer as more activities become extant in the world economy. Hence the dynamic evolutionary process has a natural interpretation involving the introduction of new goods from a finite pre-determined set all possible goods.

Lemma 7 *Let U be a limit set of the imitation dynamics, and consider two goods $i < j$. Assume that $i \in I^F(U) \backslash I^H(U)$ and $j \in I^H(U) \cap I^F(U)$. Then a single mutation suffices to have the unperturbed process enter with positive probability into a limit set U' with $i \in I^H(U')$ and either $j \notin I^H(U')$ or $i \notin$*

$I^F(U')$. *Conversely, if* $j \in I^H(U) \backslash I^F(U)$ *and* $i \in I^H(U) \cap I^F(U)$, *the same conclusion applies to a limit set* U' *with* $j \in I^F(U')$ *and either* $i \notin I^H(U')$ *or* $j \notin I^F(U')$.

Proof. Consider the first case, and posit that $i \in I^F(U) \backslash I^H(U)$ and $j \in I^H(U) \cap I^F(U)$ for some limit set U. Let $z \in U$. Since both good i and good j are produced in the foreign country, the following approximate equality must hold:

$$\frac{p_i(z)}{p_j(z)} \approx \frac{a_i^F}{a_j^F}.$$

Since the chain of comparative advantage (1) implies that $a_i^H / a_j^H < a_i^F / a_j^F$, a domestic firm that mutates and produces good i has higher fitness than that obtained in other domestic sectors (approximately $p_j(z)/a_j^H$). This implies that there is positive probability that the ensuing limit set U' at which the process arrives through the operation of the imitation dynamics has $i \in I^H(U')$. The further conclusion that either $j \notin I^H(U')$ or $i \notin I^F(U')$ is a direct consequence of Lemma 5. The second case in the statement of the result is symmetric. ■

Lemma 7 shows that even the case of incomplete specialization respects the chain of comparative advantage. Building upon these four lemmata, we can now prove that the coradius of the Walrasian limit set is unity. In essence, what remains to show is that a chain of single mutations may lead the process to a situation where the marginal good separating the specialization patterns of each country is the good with index $\tilde{k}$, just as in the Walrasian equilibrium of the benchmark economy.

Proposition 1 *Under our maintained assumptions on preferences, if the world economy is large enough, the modified coradius of* $\tilde{\Omega}$ *is 1.*

Proof. In view of the last four Lemmata, it is clear that, given any U_0, one can rely on single mutations to construct a chain of limit sets, $U_0, U_1, ..., U_r$, such that

$$\begin{aligned} I^H(U_r) &\subset \{1, 2, ..., \hat{k}\} \\ I^F(U_r) &\subset \{\hat{k}, \hat{k}+1, ..., \ell\} \\ I^H(U_r) \cup I^F(U_r) &= \{1, 2, ..., \ell\}. \end{aligned}$$

for or some $\hat{k}$. If $\hat{k} = \tilde{k}$ described in (4) and (5), the desired conclusion follows, since $U_r \subset \tilde{\Omega}$. In that case, for any $z \in U_r$, the following two conditions must jointly hold:[5]

$$\frac{a^H_{\hat{k}-1}}{a^H_{\hat{k}}} \lesssim \frac{p_{\hat{k}-1}(z)}{p_{\hat{k}}(z)} < \frac{a^F_{\hat{k}-1}}{a^F_{\hat{k}}} \tag{18}$$

$$\frac{a^H_{\hat{k}}}{a^H_{\hat{k}+1}} < \frac{p_{\hat{k}}(z)}{p_{\hat{k}+1}(z)} \lesssim \frac{a^F_{\hat{k}}}{a^F_{\hat{k}+1}}. \tag{19}$$

Otherwise, if $\hat{k} \neq \tilde{k}$, at least one of the above conditions must be violated. For concreteness, we will focus on (18), but the situation where (19) is violated is completely analogous.

There are two possibilities. The first is

$$\frac{a^H_{\hat{k}-1}}{a^H_{\hat{k}}} > \frac{p_{\hat{k}-1}(z)}{p_{\hat{k}}(z)}. \tag{20}$$

Then, since U_r is a limit set, good $\hat{k}$ is *not* an active sector in the home country. Hence, $\hat{k} \notin I^H(U_q)$. But if this fact is true, then (20) implies that a domestic mutation into producing good $\hat{k}$ in the country will prosper. Such a mutation increases the range of specialization of the home country, possibly reduces that of the foreign country, and brings the process closer to the set $\tilde{\Omega}$.

Consider now the second way to violate (18)

$$\frac{p_{\hat{k}-1}(z)}{p_{\hat{k}}(z)} \gtrsim \frac{a^F_{\hat{k}-1}}{a^F_{\hat{k}}}.$$

Now the chain of comparative advantage (1) implies that $p_{\hat{k}-1}(z)/p_{\hat{k}}(z) > a^H_{\hat{k}-1}/a^H_{\hat{k}}$. Hence if a foreign firm mutates into producing good $\hat{k}-1$, it will prosper, and this mutation leads to a new limit set with $I^F(\cdot) = \{\hat{k}-1, \hat{k}, ..., \ell\}$. Again, this brings the process closer to the Walrasian limit set $\tilde{\Omega}$. Eventually this set will be reached. We have thus established that for any initial set U_0 there is a chain of limit sets $U_0, U_1, ..., U_r$ with the two requisite features: (1) each is separated by one mutation from the next; and (2) the last one is $\tilde{\Omega}$.

The existence of such a chain readily implies, as desired, that $\widehat{CR}(\tilde{\Omega}) = 1$. To see this, simply recall that, at each step in the chain, the transition from U_{i-1} to U_i involves one mutation while for each U_i, $i = 0, 1, ..., r-1$, its radius $R(U_i) = 1$. Thus, using the definition of modified cost $\hat{c}$ given in (14), we have:

$$\max_{z \notin U} \min_{z' \in U} \hat{c}(z, z') = 1.$$

This completes the proof of the proposition. ■

[5] Here we use the notation $\lesssim$ to indicate an approximate inequality. As we have explained, the need to contemplate this approximation occurs because the number of firms is finite. However, this approximation is arbitrarily accurate as the number of firms becomes large.

We can now state and prove the main contribution of this paper. In particular, we show that Ricardo's (1817, Chapter 7) insight is robust in the face of evolutionary pressures.

Theorem 3 *Assume that preferences are homothetic and representable by a smooth, increasing, strictly quasi-concave, and additively separable utility function. Then if the world economy is large enough, the set of stochastically stable states satisfies* $\Omega^* \subset M_q(z^*)$. *Thus for all* $z \in \Omega^*$ *production follows the chain of comparative advantage.*

Proof. In view of Proposition 1 and Theorem 2, it is enough to show that $R(\tilde{\Omega}) > 1$. Recalling that $I^H(\tilde{\Omega}) = \tilde{I}^H$ and $I^F(\tilde{\Omega}) = \tilde{I}^F$ are the ranges of specialization defined in (4) and (5). Then the fact that $R(\tilde{\Omega}) > 1$ is a direct consequence of the following equilibrium conditions at any $z \in \tilde{\Omega}$:

$$\begin{aligned} \forall i \in \tilde{I}^H,\ \forall j \notin \tilde{I}^H, \qquad \frac{a_i^H}{a_j^H} &< \frac{p_i(z)}{p_j(z)} \\ \forall i' \in \tilde{I}^F,\ \forall j' \notin \tilde{I}^F, \qquad \frac{a_{i'}^F}{a_{j'}^F} &< \frac{p_{i'}(z)}{p_{j'}(z)}. \end{aligned}$$

Therefore, the following two conclusions apply. First, if any home firm mutates at z from producing $i \in \tilde{I}^H$ to producing a good $j \notin \tilde{I}^H$, its fitness (approximately $p_j(z)/a_j^H$) is lower than the fitness earned in all the other sectors $i \in \tilde{I}^H$ (approximately $p_i(z)/a_i^H$). Second, if any foreign firm mutates at z from producing $i' \in \tilde{I}^F$ to producing a good $j' \notin \tilde{I}^F$, its fitness (approximately $p_{j'}(z)/a_{j'}^F$) is lower than the fitness earned in other sectors $i' \in \tilde{I}^F$ (approximately $p_{i'}(z)/a_{i'}^F$).

Thus more than one mutation is needed for the process to escape $\tilde{\Omega}$ with positive probability. The proof of the theorem is complete. ■

Theorem 3 is the main contribution of our work, and it conveys a number of important points. First, it establishes that production according to comparative advantage has very sound evolutionary foundations. So we find that Alchian's (1950) insight was right on the mark for this model, and Fisher and Kakkar's (forthcoming) analysis of comparative advantage in a matching model with two goods can be extended significantly to a framework where prices matter. Second, Theorem 3 represents a significant contribution to the evolutionary foundations of general equilibrium theory. It shows that the main features of a classic model of general equilibrium with production are robust to an extraordinarily simple specification of the decision rules of the agents in the economy. It is worth mentioning that our analysis provides an analytically tractable model of imperfect competition in general equilibrium; the Nash equilibrium of a game whose strategic choice was which sector to enter is closely related to the long-run equilibria of our adjustment process. Third, Theorem 3 proves that the chain of comparative advantage has an elegant structure in which step-by-step evolution – including an interpretation of the introduction of new goods – has an important role. This last result implies, in particular, that evolution actually occurs fairly rapidly. We now turn our attention to this important fact.

4 How long is the long run?

Having established that production according to the chain of comparative advantage is the natural outcome of an evolutionary process, we ask the following obvious question: How long might be expected to take for this state of affairs to arise? In this section, we show that the maximum expected waiting time is independent of the size of the economy and that it is actually quite short in evolutionary terms.

4.1 Expected waiting times

In stochastic evolutionary models, it is a legitimate concern that the long-run behavior of the process might only materialize at too slow a rate. An equivalent objection is that the expected waiting times before the process reaches a stochastically stable state are too long to be really meaningful.

Ellison (2000) suggests a useful way to assess this issue. In particular, he shows that the coradius of a union of limit sets of the unperturbed dynamics can be used to provide an *upper bound* on those magnitudes; in particular, he computes the maximum expected waiting time to reach those limit sets. For example, since it is clear that seeding all the right sectors in the world economy suffices to enter the basin of attraction of $\tilde{\Omega} = \cup_{i=1}^{n} L_i$, its coradius is bounded above by $\ell + 1$. Using Ellison's results (his Theorem 1), this allows us to assert that, for any given mutation rate ε, the maximum expected waiting time to reach some state in $\tilde{\Omega}$ from any other state in Z, $W_\varepsilon(\tilde{\Omega})$, is of order no higher than $\varepsilon^{-(\ell+1)}$. That is, there exists some $C > 0$, $\bar{\varepsilon} > 0$ such that

$$\forall \varepsilon \in (0, \bar{\varepsilon}), \qquad W_\varepsilon(\tilde{\Omega}) < C \frac{1}{\varepsilon^{\ell+1}}. \tag{21}$$

The important implication of this conclusion is that $W_\varepsilon(\tilde{\Omega})$ can be bounded above, for small ε, independently of the population size ($N = N^H + N^F$). Thus, the qualitative long-run predictions of the model are to be conceived as applying essentially unchanged for both small and large economies.

But, in fact, Ellison (2000) shows that an analogous but generally much stronger conclusion applies in terms of the modified coradius. (his Theorem 2.) Since, as we have shown, the modified coradius of $\tilde{\Omega}$ is unity, the counterpart of (21) reads:

$$\forall \varepsilon \in (0, \bar{\varepsilon}), \qquad W_\varepsilon(\tilde{\Omega}) < C \frac{1}{\varepsilon},$$

which indicates that the maximum expected waiting time can be bound not only independently of population size but also of any other parameter of the model.[6] The force of this conclusion is illustrated in the examples below.

[6] One word of caution should be interjected here; the constant C may well depend upon

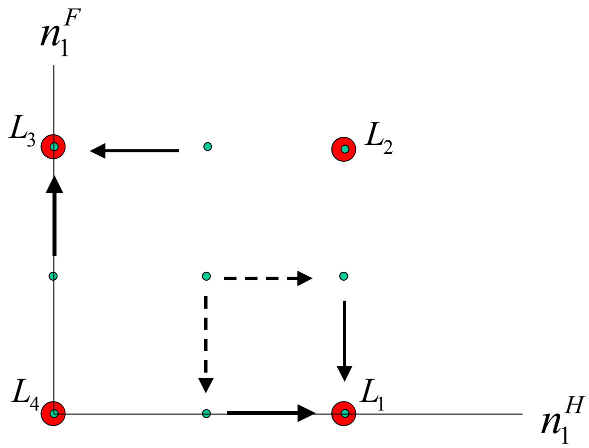

Figure 1: State space and posssible transitions for Example 1

4.2 Some illustrative examples

Example 1 *Let $\ell = 2$, $a^H = (1,2)'$, $a^F = (2,1)'$. Also, let $N^H = 2$ and $N^F = 2$. Describe preferences by $u(x) = \sqrt{x_1}\sqrt{x_2}$. This simple specification allows us to represent the state space as a two-dimensional lattice. Its nine points are shown in Figure 1. The lower left point corresponds to both firms producing none of good 1, and its upper right one corresponds to both firms producing only good 1. Figure 1 also highlights the singleton limit sets of the imitation dynamics, $L_1, ..., L_4$. Production according to comparative advantage occurs in L_1, and the "wrong" pattern of specialization occurs at L_3. The basin of attraction for L_1 consist of (1,1) and the points to the southeast of it. The solid arrows show transitions between states that occur with probability one, while dashed arrows indicate transitions that occur with (positive) probability less than unity. The radius of L_1 is 2 since it takes at least two mutations to leave that limit set. Its modified coradius is also 2 since the system cannot leave L_3 unless two or more mutations occur. Thus, in the present case, we can no longer be sure that all stochastically stable states display a production pattern according to comparative advantage. In fact, one may rely on the graph-theoretic techniques used by Kandori, Mailath, and Rob (1993) and Young (1993) to show that the set of stochastically stable states in this case is $\Omega^* = L_1 \cup L_3$.*

Our first example serves as a reminder that Ellison's approach (e.g. Theorem 2) only gives sufficient conditions for a union of limits set to be in the stochastically stable set and thus may prove inadequate in some cases to attain a full characterization of Ω^*. It also shows why our arguments depend upon having a sufficiently large world economy.

the number of goods ℓ and the ratio N^H/N^F.

The second example shows that a very large economy is not necessary for production according to comparative advantage to emerge in the long run

Example 2 *Let preferences and production parameters be the same as in the previous example, but now set* $N^H = 3$ *and* $N^F = 3$. *Then the state space has sixteen points. The singleton limit sets of the imitation dynamics are again the four corners of the lattice, and we can label them as before. The basin of attraction for* L_1 *consist of the nine points to the southwest of (1,2). The radius of* L_1 *is 3 since it takes at least three mutations to escape production according to comparative advantage. On the other hand, the (ordinary) coradius of* L_1 *is 2, since the system can leave* L_2, L_3, *or* L_4 *and enter the basin of attraction of* L_1 *through the mutation of one domestic and one foreign firm. Thus, since* $R(L_1) > CR(L_1)$, $\Omega^* = L_1$ *and the unique stochastically stable state now displays a pattern of production according to comparative advantage.*

The next example is a textbook case of comparative advantage with one large and one small country. It might seem that having the pattern of production concentrated the wrong way would be an insurmountable obstacle to production according to comparative advantage, but the example turns that intuition on its head. In particular, once a firm in the large country experiments by chance with producing the right good, then a whole cascade of firms in that country turns the pattern of trade around.

Example 3 *Let* $\ell = 2$, $a^H = (1,1)'$, $a^F = (2,1)'$. *Also, let* $N^H = 100$ *and* $N^F = 5$. *Describe preferences by* $u(x) = \sqrt{x_1}\sqrt{x_2}$. *Then the Walrasian equilibrium has* $p^* = (1,1)'$ *and* $y^* = (52.5, 52.5)$. *The home country produces the two goods, and the foreign country produces only the second one. The equilibrium allocations have the representative agent at home consume* $20y^*/21$ *and the representative foreign agent* $y^*/21$.

Consider the set $L^* = \{z_1, z_2\} = \{(53, 47; 0, 5)', (52, 48; 0, 5)'\}$, *which is a limit set of the imitation dynamics cycling around the Walrasian allocation. Within this set, the corresponding prices are* $p(z_1) = (1/53, 1/52)'$, *and* $p(z_2) = (1/52, 1/53)'$. *We argue that* $\Omega^* = L^*$, *i.e.* z_1 *and* z_2 *are the only stochastically stable states.*

The radius of L^* *is* $R(L^*) = 5$ *because it takes five mutations to shut down foreign production in the second sector. The maximum number of mutations needed to enter the basin of attraction* $A(L^*)$ *occurs when* $z_0 = (0, 100; 5, 0)'$, *and production is opposite to the pattern of comparative advantage. Prices in the first sector are so high that any foreign mutant producing in the second sector would not survive. Still, only two mutations are required to enter the basin of attraction* $A(L^*)$ *from* z_0 . *First, one home firm has to experiment producing good 1; this mutation will cause a cascade of home firms into the first sector. Then, one foreign firm has to experiment producing good 2. The intermediate limit set in this transition is* $\tilde{L} = \{(48, 52; 5, 0), (49, 51; 5, 0)\}$, *where firms in the domestic economy cycle back and forth, "waiting" for some foreign firms to mutate into the second sector. Notice that* $R(\tilde{L}) = 1$ *because it requires* *only one foreign mutation to leave this limit set. The modified coradius of* L^* *thus is* $\widehat{CR}(L^*) = 2 - 1 = 1 < 5 = R(L^*)$, *and we may therefore conclude that* $\Omega^* = L^* = \{z_1, z_2\}$. *In both* z_1 *and* z_2, *countries produce according to comparative advantage.*

Even in the worst-case scenario (i.e. starting at z_0*), the expected waiting time to enter* Ω^* *is of order* $1/\varepsilon$. *Notice how step-by-step evolution helps the world economy achieve efficiency here. The quickest path into the basin of attraction* $A(\Omega^*)$ *involves the large (domestic) country creating a market in both goods, and then the small (foreign) country "learning" to produce according to comparative advantage.*

The number of firms involved in the previous example is a bit generous. We show in the following example that *step-by-step evolution* can also play a crucial role in attaining specialization according to comparative advantage within an economy with relatively few firms.

Example 4 *Let* $\ell = 3$, $a^H = (1,1,1)'$, $a^F = (3,2,1)'$. *Also, let* $N^H = 5$ *and* $N^F = 5$. *Describe preferences by* $u(x) = (x_1)^{1/3}(x_2)^{1/3}(x_3)^{1/3}$. *Then the Walrasian equilibrium has* $p^* = (2,2,1)'$ *and* $y^* = (5/2, 5/2, 5)$. *The home country produces the first two goods, and the foreign country produces the third one. The equilibrium allocations have the representative agent at home consuming* $2y^*/3$ *and the representative foreign agent* $y^*/3$.

Consider the set $L^* = \{z_1, z_2\} = \{(3,2,0;0,0,5)', (2,3,0;0,0,5)'\}$, *which is a limit set around the Walrasian allocation. Within* L^*, *the corresponding prices are* $p(z_1) = (1/3, 1/2, 1/5)'$ *and* $p(z_2) = (1/2, 1/3, 1/5)'$. *We claim that* $\Omega^* = L^*$, *i.e. the stochastically stable states are* z_1 *and* z_2.

The radius of the limit set L^* *is* $R(L^*) = 2$ *since two mutations can shut down either active sector in the home economy. On the other hand, it should be clear that the ordinary coradius of* L^* *is* $CR(L^*) \geq 3$ *since a transition to it from the singleton limit set* $L_0 = \{(0,0,5;5,0,0)'\}$ *– where production is opposite the pattern of comparative advantage – must involve at least three mutations. Thus, we cannot invoke Theorem 1 and must instead rely on the possibilities afforded by step-by-step evolution, as described by Theorem 2.*

First note that a single mutation by a domestic firm entering the second sector allows the process to go from L_0 *into the limit set* $L_1 = \{(0,2,3;5,0,0)', (0,3,2;5,0,0)'\}$. *Another mutation by a foreign firm entering the third sector leads to a path* $(z_1, z_2, ..., z_6)$ *in the imitation dynamics as follows:*

$$\begin{aligned} z_1 &= (0,3,2;5,0,0)' \\ z_2 &= (0,3,2;4,0,1)' \\ z_3 &= (0,3,2;3,0,2)' \\ z_4 &= (0,4,1;3,0,2)' \\ z_5 &= (0,4,1;2,0,3)' \\ z_6 &= (0,5,0;3,0,2)'. \end{aligned}$$

The state z_6 *is in the limit set* $L_2 = \{(0,5,0;3,0,2)', (0,5,0;4,0,1)'\}$. *Finally, one more mutation by a domestic firm into the first sector drives all the foreign* *firms out of that sector by a path* $(z_1', z_2', ..., z_6')$ *of the imitation dynamics as follows:*

$$
\begin{aligned}
z_1' &= (0,5,0;3,0,2)' \\
z_2' &= (1,4,0;3,0,2)' \\
z_3' &= (2,3,0;3,0,2)' \\
z_4' &= (2,3,0;2,0,3)' \\
z_5' &= (2,3,0;1,0,4)' \\
z_6' &= (2,3,0;0,0,5)'
\end{aligned}
$$

where $z_6' \in \Omega^*$. *Note that*

$$R(L_1) = R(L_2) = 1. \tag{22}$$

Thus we have constructed a path that leads into $A(L^*)$ *with the minimum number of mutations. The total cost associated with this path attains the minimum value of* 3. *However, in view of (22), the modified cost of this path is just one, which readily implies that the modified coradius of* L^* *is* $\widehat{CR}(L^*) = 1 < 2 = R(L^*)$. *Therefore, we conclude that* $\Omega^* = L^*$, *as claimed.*

As before, if the system starts outside Ω^*, *the maximum expected waiting time to enter* Ω^* *is of order* $1/\varepsilon$. *An important feature of this example is that none of the intermediate limit sets involves an activity that has a large basin of attraction. This is why preferences matter. In this case, it was easy to open new sectors and thus allow a full price system to become operational. Then the pattern of production according to comparative advantage can emerge endogenously.*

4.3 Some computer simulations

We conclude this section with some simulations of the economies described in Examples 3 and 4.[7] The first column in Table 1 reports simulations of the economy described in Example 3, whereas the second column reports simulations of the economy in Example 4. Each cell reports the minimum, median, and maximum waiting times to enter the set of stochastically stable states, Ω^*, for each of these two economies. We ran 100 independent runs of 1000 periods for each scenario, all starting at the "worst state," i.e. $z_0 = (0, 100; 5, 0)$ for economy 1 and $z_0 = (0, 0, 5; 5, 0, 0)'$ for economy 2. The arguments in subsection 4.1 led us to expect that the waiting times increase rather slowly as the mutation rate falls, and the simulations confirm this fact. This aspect of the model lend credence to the relevance of stochastic stability as a criterion for long-run selection, even for large and complex economies consisting of many firms and sectors.

[7] These simulations were written in Gauss programming language. Send email to fisher.244@osu if you want a free copy of the code.

Table 1: Minimum, Median, Maximum Waiting Times to Enter the Walrasian Limit Set Ω^*

	Economy 1	*Economy 2*
$\varepsilon = .1$	$(74, 270, 741)$	$(22, 65.5, 300)$
$\varepsilon = .08$	$(93, 273.5, 784)$	$(22, 72, 252)$
$\varepsilon = .06$	$(96, 241.5, 756)$	$(26, 71, 237)$
$\varepsilon = .04$	$(84, 266, 772)$	$(18, 92, 396)$
$\varepsilon = .02$	$(88, 265.5, 891)$	$(30, 155.5, 949)$

Note: Each cell reports statistics from 100 separate runs, each consisting of 1000 periods.

5 Conclusion

We have shown that production according to the chain of comparative advantage is the outcome of a very simple evolutionary model. This result is an important confirmation of the insights of both Ricardo and Alchian. In a model that uses almost none of the precepts of *homo economicus*, the chain of comparative advantage arises naturally. This is a startling aspect of equilibrium in an economy with production, and it is largely independent of the specification of preferences in the world economy. That it can arise robustly in a model in which there is almost no explicit sense of conscious decision making or profit maximization shows that it is a deep part of the structure of economic life.

We also showed that the chain of comparative advantage allows for step-by-step evolution in sufficiently large economies. We believe that this is one of the first demonstrations of the importance of this aspect of evolution in a simple and compelling economic model. It is a very powerful result that the chain of comparative advantage will arise fairly quickly, even in a world economy with many different potential activities.

The model imposed the assumption of identical homothetic additively separable preferences, and this may have been a bit extreme. This assumption allowed us to exploit the fact that there was a unique equilibrium for the world economy and also that it had attractive stability properties. The assumption of identical preferences was too strong, and we are fairly certain that most of our ideas will generalize to an economy where preferences were allowed to be dissimilar, as long as all the agents in each economy agreed on the same measure of local fitness for each of the potential activities in the world economy.

Also, we looked at a model with only two countries. That assumption is obviously restrictive, but it allows one to write down the chain of comparative advantage. And, as we have seen, that chain has a very elegant interpretation to the evolutionary theorist. A model with several countries is obviously more realistic, but we are not sure what the speed of convergence to the Walrasian equilibrium would be. We conjecture with some conviction that production according to multi-dimensional comparative advantage would still be the outcome of an evolutionary process.

We hope that others will share our interest in the evolutionary underpinnings of the theory of general equilibrium. We have chosen to focus on a model of international trade because that field has long allowed fertile applications of general equilibrium theory. Applications of evolutionary models seem to hold promise for future research in public finance, industrial organization, labor economics, and other fields.

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