1 Problem Set I Cournot.

1. (EXCESS ENTRY) (Vives 4.12) Consider a Cournot market with linear demand $P = a - bQ$, $a > 0$, $b > 0$ and zero marginal costs for all the firms. Find the free-entry equilibrium if there is a fixed cost of entry F. Compare the free-entry equilibrium with the optimal number of firms to maximize social welfare.

2. (UNICITY) Prove that equilibrium is unique in a Cournot market whose demand is given by $P(Q)$ and the cost function of firms is given by $C_i(q_i)$ $i = 1...n$. if the following conditions hold:

- $P'(Q) < 0$ and $\bar{Q}$ exists such that for $Q \geq \bar{Q}$ we have that $P(Q) = 0$

- $P(Q)$ is log-concave. (A function f is log-concave (log-convex) if $\log f$ is concave (convex)).

- $C_i"(q_i) - P'(Q) > 0$ for all i.

(Hint: check first of all that the profit function of a firm is quasi-concave on its output).

3. (Corchón (1994)) Assume that market demand is given by $P = a - bQ$ and cost of firms by $C(q_i) = cq_i + \frac{d}{2}q_i^2$ with $a > c$, $d < 0$, $d + 2b > 0$ and $d + b < 0$. (Assume that $\dfrac{a-c}{2b+d} < -\dfrac{c}{d}$ to avoid negative costs). Show that total output decreases with entry.

4. Suppose that costs are zero and that market demand is given by:

$$P(Q) = 100E^{-aQ}$$

Show that the reaction function of firms have zero slope and find the equilibrium with n firms. Check that demand is log-convex and log-concave.

5. (Martin (2001) p. 351) Assume we have n firms competing à la Cournot with demand given by $P(Q)$ satisfying:

$$P'(Q) + QP''(Q) < 0$$

Assume that all firms have zero costs except firm 1 that has unit cost $c(k)$, where $c'(k) < 0$. Obtain the marginal change in the equilibrium outputs of firms due to a marginal increase in k.

6. (Moulin (1986) p.118) Suppose that n firms compete à la Cournot in a market whose concave demand is given by $P(Q)$. Each firms has an increasing and convex cost function $c_i(q_i)$. Show that in the Cournot equilibrium industry profits are not maximized.

7. Suppose we have two symmetric firms competing à la Cournot in a market with demand $P = 1 - Q$. Production costs of firms are given by $C(q) = F$ if $q > 0$ and $C(0) = 0$. Find the Cournot equilibria depending on the value of F.

8. Assume that market demand $P(X)$ is concave and strictly decreasing. We have n firms, denoted by a natural number from 1 to n. Firm i's cost function $C_i(x_i)$ is strictly convex and $C_i'(0) = 0$. Assume that in the unique Cournot equilibrium and in the allocation that maximizes social welfare all firms produce a positive quantity. Then show that the quantity sold at the (unique) Cournot

equilibrium is lower than the quantity sold in the allocation that maximizes social welfare.

2 Solution to Problem Set I:

1. the number of firms with free entry $n^{FE} = a\sqrt{\frac{1}{bF}} - 1$.

the number of firms that maximizes social welfare: $n^* = (\frac{a^2}{bF})^{1/3} - 1$

If a monopoly is profitable ($\frac{a^2}{4b} > F$), then $\frac{a^2}{bF} > 1$ and therefore $n^{FE} > n^*$.

2. The fact that $P(Q)$ is log-concave implies that:

$$P"P - (P')^2 \leq 0 \tag{1}$$

We assume furthermore that:

$$C" - P' > 0. \tag{2}$$

The FOC of firm i is given by:

$$\frac{\partial \pi_i}{\partial x_i} = P(Q) + q_i P'(Q) - C_i'(q_i) = 0 \tag{3}$$

Second order conditions are satisfied, because whenever (7) holds, the function is strictly concave, which implies that π_i is strictly quasi-concave with respect to q_i. To see this we subdivide the Second Order Condition in two parts:

$$\frac{\partial^2 \pi_i}{(\partial x_i)^2} = [P'(Q) + q_i P"(Q)] + \left[P'(Q) - C_i^{"}(q_i)\right]$$

The second part is negative. We are going to see that the second part is not positive. If the function is concave we are done. Then we have to check it for the case of convex demands. From (3) we have that:

$$q_i = -\frac{P - C_i'}{P'}.$$

Then;

$$\begin{aligned} P'(Q) + q_i P"(Q) &= P'(Q) + (-\frac{P - C_i'}{P'})P"(Q) = \\ -\frac{1}{P'}(-(P')^2 + (P - C_i')P") &< -\frac{1}{P'}(-(P')^2 + PP") \leq 0 \end{aligned}$$

The first inequality cames from increasing costs and demand convex and the second from (1). Quasi-concavity of profits imply that a Nash equilibrium exists (Theorem II.1 in Vives).

The fact that π_i is strictly quasi-concave with respect to q_i, implies that there is a unique profit-maximizing strategy of firm i, defined by (3) and denoted by

$r_i(q_{-i})$. In other words, the reaction functions are well-defined. The proof of unicity comes from the fact that the slope of the reaction function of firms is negative and greater than -1. Applying the implicit function theorem to (3) we have that:

$$r_i'(q_{-i}) = -\frac{P'(Q) + q_i P"(Q)}{2P'(Q) + q_i P"(Q) - C_i"(q_i)}$$

We have checked that both the numerator and the denominator are negative and given (2), the denominator is bigger in absolute value than the numerator. Then $-1 < r_i'(q_{-i}) < 0$.

Define the cumulative reaction function $\phi_i(Q)$ from:

$$\begin{aligned} q_i &= r_i(Q - q_i) \\ \phi_i'(Q) &= \frac{r_i'(x_{-i})}{1 + r_i'(x_{-i})} < 0 \end{aligned}$$

$\phi_i(Q)$ is defined over $[r_i(0), \bar{Q}]$. Then the equilibrium output will be given by the fixed point of $\Phi(Q) = \sum_{i=1}^{n} \phi_i(Q)$ and the individual outputs by $\phi_i(Q)$. $\Phi(Q)$ is defined over $[\max_i r_i(0), \bar{Q}]$. Existence and unicity follow from the fact that $\Phi(Q)$ is continuous and decreasing and that $\Phi(\max_i r_i(0)) \geq \max_i r_i(0)$ and $\Phi(\bar{Q}) = 0$.

3. Total output in equilibrium is given by: $Q = \dfrac{n(a-c)}{(n+1)b+d}$ $.\dfrac{\partial Q}{\partial n} = \dfrac{(a-c)(b+d)}{((n+1)b+d)^2} < 0$.

4. $R_i(q_{-i}) = \dfrac{1}{a}$. In equilibrium each firm produces $\dfrac{1}{a}$

$\log(P(Q)) = -aQ$ is a linear function. Then, $P(Q)$ is log-concave and log-convex.

5. The Cournot equilibrium is given by the solution of the system of the F.O.C.

$$\begin{aligned} P(Q) + q_1 P'(Q) - c(k) &= 0 \\ P(Q) + q_i P'(Q) &= 0 \; i = 2, ...n \end{aligned} \tag{4}$$

Taking the the total differential of (4) we have:

$$\begin{aligned} dq_1 &= -\mu_1 dQ + \chi_1 dk \\ dq_i &= -\mu_i dQ \\ i &= 2...n \\ \mu_i &= \frac{P'(Q) + q_i P''(Q)}{P'(Q)} \text{ and } \chi_1 = \frac{c'(k)}{P'(Q)} \end{aligned} \tag{5}$$

Adding (5) for all i we have:

$$\begin{aligned} dQ &= -\mu dQ + \chi_1 dk \qquad (6) \\ \frac{dQ}{dk} &= \frac{\chi_1}{1+\mu} \\ \text{where } \mu &= \sum_{i=1}^{n} \mu_i \end{aligned}$$

Using (5) and (6) we have:

$$\begin{aligned} \frac{dq_1}{dk} &= \chi_1 (1 - \frac{\mu_1}{1+\mu}) \\ \frac{dq_i}{dk} &= -\frac{\mu_i \chi_1}{1+\mu} \end{aligned}$$

6. The Cournot equilibrium is determined by the first-order conditions:

$$\begin{aligned} q_i^* P'(Q^*) + P(Q^*) &= c_i^{'}(q_i^*) \\ i &= 1...n \end{aligned}$$

Joint profit is given by:

$$\pi(q_1,...q_n) = QP(Q) - \sum_{i=1}^{n} c_i(q_i)$$

$$\frac{\partial \pi}{\partial q_i}(q_1,...q_n) = P(Q) + QP'(Q) - c_i'(q_i)$$

Evaluated at the Cournot equilibrium we have:

$$\frac{\partial \pi}{\partial q_i}(q_1^*,...q_n^*) = (Q^* - q_i^*)P'(Q^*) < 0$$

So reducing any one of the individual productions increases joint profit.

7. Call one firm, firm 1 and the other firm 2.

The reaction function of firms is $r_i(q_j) = \frac{1-q_j}{2}$ if $q_j \leq 1 - 2\sqrt{F}$ and $r_i(q_j) = 0$ otherwise. $(i, j = 1, 2 \ i \neq j)$. We have a Cournot equilibria wherever reaction functions cross. Then we distinguish different cases. They can be understood taking into account the following. In the standard Cournot equilibrium $(F = 0)$, profits of firms are $\frac{1}{9}$. It will still be an equilibrium if $F \leq \frac{1}{9}$. The other equilibrium that may arise is that one firm produces the monopoly output $(\frac{1}{2})$ and the other firm does not want to produce. This will be the case if $\frac{1}{16} \leq F$, because

$$\underset{q}{Max}(1 - q - \frac{1}{2})q = \frac{1}{16}$$

. Finally when the fixed costs are greater that the monopoly profits $\frac{1}{4}$, no firm wants to produce.

Case 1: If $1-2\sqrt{F} > \frac{1}{2}$ or $0 \leq F < \frac{1}{16}$.

Reaction functions cross only once at $q_1 = q_2 = \frac{1}{3}$.

Case 2: If $\frac{1}{3} \leq 1-2\sqrt{F} \leq \frac{1}{2}$ or $\frac{1}{16} \leq F \leq \frac{1}{9}$.

Reaction functions cross three times at $(q_1 = q_2 = \frac{1}{3})$, $(q_1 = \frac{1}{2},\, q_2 = 0)$ and $(q_1 = 0,\, q_2 = \frac{1}{2})$.

Case 3: $0 \leq 1-2\sqrt{F} < \frac{1}{3}$ or $\frac{1}{9} < F \leq \frac{1}{4}$.

Reaction functions cross twice at $(q_1 = \frac{1}{2},\, q_2 = 0)$ and $(q_1 = 0,\, q_2 = \frac{1}{2})$.

Case 4: $1-2\sqrt{F} < 0$ or $F > \frac{1}{4}$.

Reaction functions cross at $(q_1 = 0,\, q_2 = 0)$.

8. Let $(x_1^*....x_n^*)$ be the Cournot equilibrium where $x^* = \sum_{i=1}^n x_i^*$.

It satisfies the F.O.C. of profit maximization:

$$P(x^*) + P'(x^*)x_i^* - C_i'(x_i^*) = 0$$

Let $(x_1^o....x_n^o)$ be the allocation that maximizes social welfare where $x^o = \sum_{i=1}^n x_i^o$. They satisfy the F.O.C. of maximizing social welfare given by

$$\int_0^{\sum_{i=1}^n x_i} P(x)dx - \sum_{i=1}^n C_i(x_i)$$

$$P(x^o) - C_i'(x_i^o) = 0$$

As demand is downward sloping we have that

$$P(x^*) - C_i'(x_i^*) > P(x^o) - C_i'(x_i^o)$$

$$C_i'(x_i^o) - C_i'(x_i^*) > P(x^o) - P(x^*) \tag{7}$$

Assume that $x^* \geq x^o$. Then (7) implies

$$C_i'(x_i^o) - C_i'(x_i^*) > 0$$

Given that marginal cost is strictly increasing we have that $x_i^o > x_i^*$ what is a contradiction with $x^* \geq x^o$. Then we must have that $x^* < x^o$.

3 Problem Set II Prices.

1. Assume that demand is given by $D(p) = a - bP$ and firms are symmetric with cost function given by $C(q) = \frac{c}{2}q^2$. Find the Bertrand equilibria as a function of the number of firms n.

2. (Tirole (1988) p. 319) Obtain the equilibrium of the game where two firms choose in a first stage capacity and compete à la Bertrand-Edgeworth in the second stage when demand is given by $p = 4 - Q$ and capacity costs 3 per unit. Assume that demand is rationed according to the surplus-maximizing rule.

3. Vives 5.3/5.4/5.7

4 Solution to Problem Set II

1. In class, we checked that no asymmetric equilibrium exists. Then in equilibrium all firms set the same P. It will be an equilibrium if no firm has incentive to deviate by setting a higher or a lower price. ($P \geq \frac{a}{b}$ can not be an equilibrium, because we have no sales and firms have incentives to deviate and set the monopoly price. Then from now on we only consider $P < \frac{a}{b}$.)

Observe that, given that all firms set P, no firms wants to set a higher price if the following condition holds:

$$\begin{aligned} (\frac{a-bP}{n})P - \frac{c}{2}(\frac{a-bP}{n})^2 &\geq 0 \\ (\frac{a-bP}{n})\left[P - \frac{c}{2}(\frac{a-bP}{n})\right] &\geq 0 \end{aligned}$$

$$\begin{aligned} P - \frac{c}{2}(\frac{a-bP}{n}) &\geq 0 \\ P &\geq \frac{ac}{2n+bc} \end{aligned}$$

Observe that, given that all firms set P, no firm wants to undercut if the following condition holds:

$$\begin{aligned} (\frac{a-bP}{n})P - \frac{c}{2}(\frac{a-bP}{n})^2 &\geq (a-bP)P - \frac{c}{2}(a-bP)^2 \\ \frac{P}{n} - \frac{c}{2n^2}(a-bP) &\geq P - \frac{c}{2}(a-bP) \\ \frac{ac(1+n^{-1})}{2+bc(1+n^{-1})} &\geq P \end{aligned}$$

Every firm charging the same price P is an equilibrium if $P \in [\frac{ac}{2n+bc}, \frac{ac(1+n^{-1})}{2+bc(1+n^{-1})}]$.

2. Monopoly profits are 4. Then no firm will choose k_i such that $4 - 3k_i > 0$, $k_i > \frac{4}{3}$. But if firms choose always $k_i \leq \frac{4}{3}$, we will have an equilibrium in pure strategies in the second stage, because $\frac{4}{3}$ is the Cournot equilibrium with zero costs and we are considering the surplus maximizing rationing rule. In those equilibria the profits are given by $(4 - k_1 - k_2)k_i$. The payoffs in the second stage will be given by $(4 - k_1 - k_2)k_i - 3k_i$ yielding the equilibrium choice of capacity $k_1 = k_2 = \frac{1}{3}$.

3. (Vives 5.3)

We prove it directly for the asymmetric case. We check that every firm charging the competitive price w_n and every firm selling its competitive supply at this price $S_j(w_n)$ is an equilibrium. Recall that by definition $D(w_n) = \sum_{j=1}^{n} S_j(w_n)$: at the competitive price demand equals supply. We are going to check that no firm has incentives to deviate either by raising or lowering the price.

a) If it raises the price, it sells nothing and it earns zero profits while in the competitive equilibrium obtains nonnegative profits.

b) If it lowers the price he will serve the whole market and its profits will be:

$$pD(p) - C(D(p))$$

But then as $w_n > p$ we have that:

$$pD(p) - C(D(p)) < w_n D(p) - C(D(p))$$

But as $S_j(w_n) = \arg\max_q w_n q - C(q)$ we have:

$$pD(p) - C(D(p)) < w_n D(p) - C(D(p)) \leq w_n S_j(w_n) - C(S_j(w_n))$$

This shows that by lowering the price, it obtains less profits.

4. (Vives 5.4)

To prove that $D_i^P(p) \leq D_i^{SM}(p)$, we have to check that $D(p_i) \sum_{j \in J} \dfrac{k_j}{D(p_j)} \leq \sum_{j \in J} k_j$. But given that demand is downward sloping and $p_j < p_i$ we have that $D(p_i) \sum_{j \in J} \dfrac{k_j}{D(p_j)} \leq D(p_i) \sum_{j \in J} \dfrac{k_j}{D(p_i)} = \sum_{j \in J} k_j$.

5. (Vives 5.7)

Region with pure-strategy equilibrium.

A) Take $P(Q) = 1 - Q$ to be the inverse demand. The only candidate equilibrium in pure strategies is the competitive price $P(nk)$. An equilibrium at zero price exists if rivals can serve the whole market at zero price.

$$P((n-1)k) = 0; (n-1)k \geq D(0); k \geq \frac{1}{n-1}$$

B) The following condition guarantees that firms do not want to deviate to set a higher price than $P(nk)$.

$$P(r((n-1)k) + (n-1)k) \leq P(nk); r((n-1)k) + (n-1)k \geq nk$$

$$r((n-1)k) \geq k; \frac{1-(n-1)k}{2} \geq k; \frac{1}{n+1} \geq k$$

Region without pure-strategy equilibrium:

$$\frac{1}{n-1} > k > \frac{1}{n+1}$$

We obtain the mixed-strategy equilibrium by construction when $k = \frac{1}{n}$. Assume that a symmteric equilibrium exists in which firms randomize over the support $[\underline{p}, \bar{p}]$ according to the (atomless) distribution function $F()$. As firms can guarantee themselves positive expected profits, $p = 0$ can not belong to the support and therefore not all capacity can be sold because $D(p) < 1$. Furthermore, even when setting the highest price $\bar{p}$, sales should be positive, then $D(\bar{p}) - \frac{n-1}{n} > 0$. Both facts imply that for all realizations of prices all firms sell to capacity except the firm that charges the highest price. Then the expected profits are given by:

$$F^{n-1}(1 - p - \frac{n-1}{n})p + (1 - F^{n-1})(\frac{p}{n})$$

If a firm charges $p = \bar{p}$, it will be the one that charges the highest price and then its expected payoff is

$$(1 - p - \frac{n-1}{n})p$$

$\bar{p}$ is the price that maximizes this expression:

$$\frac{1}{2n} = \bar{p} = \arg\max(1 - p - \frac{n-1}{n})p$$

Then the expected profits are given by:

$$\bar{\pi} = \frac{1}{4n^2}$$

Then the $\underline{p}$ should satisfy

$$\underline{p} = n\bar{\pi} = \frac{1}{4n}$$

The distribution function satisfies: :

$$\begin{aligned} F^{n-1}(\frac{1}{n} - p)p + (1 - F^{n-1})(\frac{p}{n}) &= \frac{1}{4n^2} \\ F(p)^{n-1} &= \frac{1}{pn} - \frac{1}{4n^2p^2} \\ F(\underline{p}) &= 0; F(\bar{p}) = 1 \end{aligned}$$

5 Problem Set III Two-stage games.

1. (STRATEGIC TRADE POLICY) Assume that n firms of country A compete with one firm of country B in a foreign market whose demand is given by $P = a - bQ$. All firms have the same unit cost c, where $a > c$. Obtain the subgame perfect equilibrium of the two-stage game in which in the first stage the Government of country A sets a unit export subsidy s that reduces the unit cost of firms of country A to $c - s$. In the second stage, firms compete à la Cournot in the foreign market. The objective of the Government of country A is to maximize the profits of their national firms net of the subsidy. Is the equilibrium export subsidy always positive ? Explain.

2. (STACKELBERG) Assume that two firms (firm 1 and 2) compete choosing prices in a differentiated goods market. The demand of the good that firm i sells is given by $q_i = 1 - p_i + cp_j$ where $0 < c < 1$ (goods are substitutes) and p_i and p_j are the prices charged by firm i and j respectively ($i, j = 1, 2$). There are no costs.

(a) Find the the equilibrium of Bertrand.

(b) Find the the subgame perfect equilibrium of the two-stage game where in the first stage firm 1(leader) chooses its price and firm 2 (follower) chooses its price in the second stage.

3. (DIVISIONALIZATION) Assume that we have two firms. Demand is given by $P = \alpha - Q$ and marginal cost by c. Find the equilibrium of the following two-stage game. In the first stage, firms decide the number of (independent) divisions, given that the cost per divisions is F. In the second stage, divisions compete a la Cournot (each division maximizes its individual profits). Assume that firms when setting the number of divisions maximize the joint profits of their divisions, for simplicity, treat the number of divisions as a continuous variable and take advantage of the fact that equilibrium is symmetric.(See Baye, M., K. Crocker, and J. Ju, 1996, Divisionalization, Franchising and Divestiture Incentives in Oligopoly, American Economic Review, 86, 223-236).

4. (Vives 6.2) (STRATEGIC MANAGERIAL INCENTIVES) Consider a Cournot duopoly with linear demand $p = a - Q$ with $a > 0$. Firms have constant marginal cost c. Suppose that each owner gives its manager an incentive contract which is a linear combination of profits (π_i) and sales (S_i):

$$\lambda_i \pi_i + (1 - \lambda_i) S_i$$

Study the two-stage game in which first owners choose λ_i and then managers, upon having observed the contracts offered to both of them, compete à la Cournot. Assume that managers are paid their (fixed) opportunity cost. Therefore when choosing λ_i owners want to maximize the profits of their firms. (See Fershtman and Judd (1987) "Equilibrium Incentives in Oligopoly" The American economic review, vol. 77 n^{o}5 pp. 927-940).

5. (ENTRY DETERRENCE) Find the Subgame Perfect Equilibrium of the following two-stage game. In the first stage, Firm 1 (leader) chooses its output while Firm 2 (follower) chooses its output in Stage 2. Variable costs are zero

but Firm 2 faces a fixed cost of F when producing. The usual interpretation is that Firm 1 is the incumbent and Firm 2 a potential entrant. Market demand is given by $P = 1 - Q$.

6. (ENTRY DETERRENCE) Find the Subgame Perfect Equilibrium of the following two-stage game. In the first stage, Firm 1 (leader) chooses its price while Firm 2 (follower) chooses its price in Stage 2. Variable costs are zero but Firm 2 faces a fixed cost of $F < \dfrac{1}{4}$ when producing. The usual interpretation is that Firm 1 is the incumbent and Firm 2 a potential entrant. Market demand is given by $D(P) = 1 - P$. Assume that when both firms set the same price all the demand goes to the follower.

7. Assume that Firm 1 and Firm 2 compete in a homogenous good market whose demand is given by $Q = 1 - P$, where P is price and Q the quantity. Unit cost of Firm 2 is given by c and the unit cost of firm 1 by $c - x_1$, where x_1 is his R+D investment. The R+D investment costs $C(x_1) = x_1^2$. Find the Subgame Perfect Nash equilibrium of the following two-stage game. Assume that $1 > c > \frac{3}{4}$.

In the first stage, Firm 1 chooses the level of its R+D investment x_1.

In the second stage, Firm 1 and 2 compete in prices. Assume that at equal prices all demand goes to Firm 1.

8. Consider a Cournot duopoly with demand $P = \dfrac{1}{Q}$. Firms have constant marginal cost equal to 1. Suppose that each owner gives its manager an incentive contract which is a linear combination of profits (π_i) and sales (S_i):

$$\begin{aligned} &\lambda_i \pi_i + (1 - \lambda_i) S_i \\ \text{where } \lambda_i \in\ &(0, 2) \end{aligned}$$

Find the Subgame Perfect Nash Equilibrium of the two-stage game in which first owners choose λ_i and then managers, upon having observed the contracts offered to both of them, compete à la Cournot. Assume that managers are paid their (fixed) opportunity cost. Therefore when choosing λ_i owners want to maximize the profits of their firms.

9. (Huck et al (2001)) Assume we have $m + n$ firms competing in a market whose demand is given by $P = \alpha - Q$. All firms have (constant) unit cost c. Find the Subgame Perfect Nash Equilibrium of the following two-stage game. In the first stage, m firms (leaders) decide their output. In the second stage, the remaining n firms (followers) decide their output. Market price is the one that equates demand with production of firms. Study the profitability of the merger of two leaders and the profitability of the merger of two followers.

6 Solution to Problem Set III

1. The output sold by the firms of country A in the equilibrium of the second stage is $q = \dfrac{a-c+2s}{b(n+2)}$ and the price margin $p - c = \dfrac{a-c-ns}{n+2}$. Then in the first stage the Government of country A maximizes $n(p-c)q$. It is maximized when $s = -\dfrac{(a-c)(n-2)}{4n}$. The same result can be obtained by equating the production of the national firms to the output of the Stackelberg leader:

$$\frac{a-c}{2b} = nq$$

2. (a) $p_1^* = p_2^* = \dfrac{1}{2-c}$.

(b)$p_1 = \dfrac{2+c}{4-2c^2}$ and $p_2 = \dfrac{4+2c-c^2}{4(2-c^2)}$.

3. $n^* = \left(\left[\dfrac{(\alpha-c)^2}{F} \right]^{(1/3)} - 1 \right) /2$

4. The quantity sold in the second stage by the manager of firm i is given by: $q_i = \dfrac{a - 2\lambda_i c + \lambda_j c}{3}$ and the price margin by $p - c = \dfrac{a + (\lambda_i - 3)c + \lambda_j c}{3}$. When choosing λ_i, the owner of firm i maximizes $(p-c)q_i$. The FOC is given then by:

$$-2c(a + (\lambda_i - 3)c + \lambda_j c) + c(a - 2\lambda_i c + \lambda_j c) = 0$$

Imposing symmtery$\lambda_i = \lambda_j = \lambda$, we obtain the equilibrium:

$$\lambda = 1 - \frac{a-c}{5}$$

5. Given an output of firm q_1, Firm 2 decides to produce $r_2(q_1) = \dfrac{1-q_1}{2}$ whenever it yields greater profits than not producing:

$$\begin{aligned} \left(\frac{1-q_1}{2}\right)^2 - F &\geq 0 \\ q_1 &\leq 1 - 2\sqrt{F} \end{aligned}$$

In the first stage, if $F \geq \frac{1}{16}$, the optimal decision of Firm 1 is easy to derive, it produces the monopoly output and Firm 2 does not enter in the second stage because $1 - 2\sqrt{\frac{1}{16}} = \frac{1}{2}$. In this situation entry is said to be blockaded. For $F < \frac{1}{16}$, the payoff of Firm 1 is given by:

$$(1 - q_1 - r_2(q_1))q_1 \text{ if } q_1 \leq 1 - 2\sqrt{F} \tag{8}$$

$$(1 - q_1)q_1 \text{ otherwise} \tag{9}$$

In (8) the optimal decision is $q_1 = \frac{1}{2}$ and profits obtained are $\frac{1}{8}$. In (9) the optimal output is the lowest output that induces that Firm 2 will not produce in the second stage. This means a profit of: $2\sqrt{F}(1-2\sqrt{F})$. To derive the optimal output one should obtain the values of F such that:

$$\begin{aligned} \frac{1}{8} - 2\sqrt{F}(1-2\sqrt{F}) &\geq 0 \\ F &\leq \frac{3-2\sqrt{2}}{32} \end{aligned}$$

Then for $F \leq \dfrac{3-2\sqrt{2}}{32}$, firm 1 will produce $q_1 = \frac{1}{2}$ and for $\dfrac{3-2\sqrt{2}}{32} < F < \frac{1}{16}$ it will produce $q_1 = 1 - 2\sqrt{F}$. In this last case, entry is said to be deterred.

6. We solve the model as usual by backward induction.

Second stage.

Firm 1 has set a price equal to p_1 in the first stage. Then we derive the optimal strategy of Firm 2.

If $p_1 \geq \frac{1}{2}$, then $p_2^* = \frac{1}{2}$. Observe that $\frac{1}{2}$ is the monopoly price.

If $p_1 < \frac{1}{2}$, then $p_2^* = p_1$ if $p_1(1-p_1) > F$ and $p_2^* = \infty$, otherwise.

First stage.

$p^* = \dfrac{1-\sqrt{1-4F}}{2} < \frac{1}{2}$ is the lowest price that satisfies $p(1-p) = F$.

If Firm 1 sets $p_1 > p^*$, he will be undercut in the second stage by Firm 2 and therefore he will obtain zero profits.

If Firm 1 sets $p_1 \leq p^*$ he will obtain profits of $p_1(1-p_1)$. This function is strictly concave maximized in $p_1 = \frac{1}{2}$. As $p^* < \frac{1}{2}$ Firm 1 will optimally choose $p_1^* = p^*$.

7. We solve the model by backward induction.

In the second stage we have Betrand competition with asymmetric costs where Firm 1 has unit cost $c - x_1$ and Firm 2 has cost c. If Firm 1 were a monopolist, it would set $p^m = \dfrac{1+c-x_1}{2}$.

Then if $p^m \leq c$ ($x_1 \geq 1-c$) the equilibrium is $p_1^* = p^m$ and $p_2^* \geq p^m$. Firm 1 obtains profits $\pi_1^m = \left(\dfrac{1-c+x_1}{2}\right)^2$.

If $p^m > c$, the equilibrium is given by $p_1^* = p_2^* = c$ and Firm 1 obtains $\pi_1 = x_1(1-c)$.

In the first stage the payoff of Firm 1 as a function of its level of investment is given by:

$$\begin{aligned} \pi_1 - x_1^2 \text{ if } x_1 &< 1-c \\ \pi_1^m - x_1^2 \text{ if } x_1 &\geq 1-c \end{aligned}$$

It is easy to verify that $\pi_1^m - x_1^2 \leq 0$ if $x_1 \geq 1-c$. Then the optimal investment will be obtained from maximizing $\pi_1 - x_1^2$.

$$\frac{\partial(\pi_1 - x_1^2)}{\partial x_1} = 1 - c - 2x_1 = 0$$

$$x_1^* = \frac{1-c}{2}$$

8. In the second stage the manager of firm i maximizes:

$$O_i = \lambda_i \left((\frac{1}{q_1+q_2} - 1) q_i \right) + (1-\lambda_i) \left(\frac{q_i}{q_1+q_2} \right) =$$

$$= \frac{q_i}{q_1+q_2} - \lambda_i q_i$$

$$\frac{\partial O_i}{\partial q_i} = \frac{q_j}{(q_1+q_2)^2} - \lambda_i = 0 \quad (10)$$

$$\frac{\partial O_1}{\partial q_1} + \frac{\partial O_2}{\partial q_2} = \frac{q_1+q_2}{(q_1+q_2)^2} - \lambda_1 - \lambda_2 = 0$$

$$q_1 + q_2 = \frac{1}{\lambda_1 + \lambda_2} \quad (11)$$

Using (11) in (10) we obtain the second stage equilibria:

$$q_i = \frac{\lambda_j}{(\lambda_1+\lambda_2)^2}$$

Then the profits of firm i as a function of the incentive parameters is:

$$\pi_i = \left(\frac{\lambda_j}{(\lambda_1+\lambda_2)^2} \right) (\lambda_1 + \lambda_2 - 1)$$

$$\frac{\partial \pi_i}{\partial \lambda_i} = \frac{-\lambda_1 - \lambda_2 + 2}{(\lambda_1+\lambda_2)^3} = 0 \quad (12)$$

It is easy to check that $\frac{\partial^2 \pi_i}{(\partial x_i)^2} < 0$ when (12) holds. Then the profit function is quasi-concave in λ_i. We have a multiplicity of equilibria that satisfy:

$$\lambda_1^* + \lambda_2^* = 2$$

$$\text{for } \lambda_i \in (0, 2)$$

In the unique symmetric equilibrium we have that $\lambda_1^* = \lambda_2^* = 1$. The incentives are not distorted from profit maximization. The reason for this is that when firms are symmetric, the reaction function has zero slope in equilibrium. Applying the implicit function theorem to (10), we have:

$$R_i^{'}(q_j) = \frac{q_i - q_j}{2q_j}$$

When the slope of the reaction function of the competitor is zero, the strategic effect of incentives disappears and the optimal contract is to tell managers to maximize profits.

9. We proceed by backward induction. Given that the leaders have produced $\sum_{i=1}^{m} q_i$, the game followers play is like a standard Cournot game with (residual) demand $P = \alpha - \sum_{i=1}^{m} q_i - Q$. Then, each follower will produce

$$q_f = \frac{\alpha - \sum_{i=1}^{m} q_i - c}{n+1} \tag{13}$$

.The profit of each leader in the first stage as a function of its output is given by:

$$\begin{aligned} \pi_i &= (\alpha - \sum_{i=1}^{m} q_i - n\left(\frac{\alpha - \sum_{i=1}^{m} q_i - c}{n+1}\right) - c)q_i = \\ &= \left(\frac{1}{n+1}\right)\left(\alpha - \sum_{i=1}^{m} q_i - c\right) q_i \end{aligned}$$

Except for a multiplicative constant, this is the same profit function as in the standard Cournot case with demand $P = \alpha - Q$ and m firms. Then both games will have the same equilibria. In equilibrium each leader produces $q_l = \dfrac{\alpha - c}{m+1}$ and then using (13) each follwer produces $q_f = \dfrac{\alpha - c}{(m+1)(n+1)}$. Then profits of leaders and followers are given respectively by $\pi_l = \left(\dfrac{1}{n+1}\right)\left(\dfrac{\alpha - c}{m+1}\right)^2$ and $\pi_f = \left(\dfrac{1}{m+1}\right)^2\left(\dfrac{\alpha - c}{n+1}\right)^2$.

Except for a multiplicative constant π_l is like the standard Cournot profits with m firms and n firms respectively. Then we can use what we know about merger profitability in the Cournot model to derive the profitabilty of mergers of leaders. In the linear Cournot model (Salant et al (1983)) mergers of two firms are never profitable except in the case of duopoly. The merger of two leaders will only be profitable if $m = 2$. For the same argument we can conclude that the merger of two followers will only be profitable if $n = 2$.

7 Problem Set IV. Integration.

1. Assume that market demand is given by $Q = 1 - P$. We have n firms with zero costs and one firm with constant unit cost c that compete à la Cournot. Suppose that one efficient firm and the inefficient firm merge and the new firm produces at zero cost.

a) Find the values of c such that the merger increases welfare. Observe that the merger may increase welfare even though both price and concentration increase. Show that the result can be stated as the merger increases welfare if

$$s < \frac{2(n+1)}{2n^2 + 6n + 5}$$

where s is the premerger market share of the inefficient firm (see Faulí-Oller (2002)).

b) Find the values of c such that the merger increases profits. Show that the result can be stated as the merger increases profits if

$$\frac{s_e}{s} > \frac{n(n+2)}{2(n+1)}$$

where s_e is the premerger market share of the efficient firm (see Faulí-Oller (2002)).

2. (DOUBLE MARGINALIZATION) Assume that an upstream firm, denoted by U, produces an input at cost c_u. n downstream firms transform the input into an homogeneous final one on a one-for-one basis and at zero marginal cost. Downstream firms compete à la Cournot. Find the (observable) price that firm U will set for the input (no fixed part is allowed). Show that its profits tend to the monopoly profits when n tends to infinity i.e. the margin of downstream firms vanishes. Market demand is given by $P = \alpha - Q$.

3. Assume that an upstream firm, denoted by U, produces an input at cost c_u. n downstream firms transform the input into an homogeneous final one on a one-for-one basis and at zero marginal cost. There also exists a competitive market where the input can be obtained at price c where $c \geq c_u$. The timing of the game is the following: in the first stage Firm U offers two-part tariffs $(w_i q + F_i)$ supply contracts to downstream firms. In the second stage they decide whether to accept or reject the contracts. Finally, downstream firms compete à la Cournot in the final market. Obtain the equilibrium payoffs of downstream firms as a function of c_u, n and c if contracts are secret and firms have passive conjectures (See Bru et al. IVIE WP-AD 2001-28). Market demand is given by $P = \alpha - Q$.

4. Assume that two firms (A and B) compete à la Cournot in a market. Market demand is given by $P = \alpha - Q$, unit production cost is c and distribution costs are zero. Firm B sells directly to consumers whereas Firm A creates independent franchises to distribute the good. Creating a franchise suppose a positive but small fixed cost ε. Analyze the following game. In the first stage, Firm A decides the number of franchises. In the second stage, it offers two

part tariff supply contracts $(w_i q + F_i)$ to the franchises. In the third stage, the franchises decide whether to accept the contract or not and in the fourth stage, independent franchises and Firm B compete a la Cournot. Obtain the Subgame Perfect Nash Equilibrium of the previous game for the following situations.

a) Supply contracts are observabe both to the franchises and Firm B.

b) Supply contracts are observable to all franchises but not to Firm B.

c) Each franchise observes only the supply contract that is offered by the franchisor, but it does not observe the contracts offered to the other franchises. Firm B does not observe any contract. Furthermore, assume that franchisees have passive conjectures.

5. Assume that an upstream firm, denoted by U, produces an input at cost $C(q) = dq^2$. One downstream firm (D) transforms the input into an homogeneous final one on a one-for-one basis and at zero marginal cost. Obtain the supply contract that U offers to D in the following situations:

a) It can offer a two-part tariff contract.

b) The supply contract can only include a linear price.

6. Assume that an upstream firm, denoted by U, produces an input at zero cost. 2 downstream firms (say Firm 1 and Firm 2) transform the input into an homogeneous final one on a one-for-one basis and at zero marginal cost. Downstream firms compete à la Cournot. In the first place, firm U sets the price for the input it charges to Firm 1 (w_1) and to Firm 2 (w_2). In the second place, firms compete a la Cournot. Find the Perfect Bayesian equilibrium if prices for the input are not observed by competitors assuming that downstream firms have passive conjectures. Market demand is given by $P = \alpha - Q$.

8 Solution to Problem Set IV.

1. Using the results obtained in class for a Cournot market with linear demand and asymmetric constant marginal costs we have that, before the merger, total output is given by $Q^B = \frac{(n+1)(1-c/(n+1))}{n+2}$, the output of the inefficient firm is $q_c^B = \frac{1-(n+1)c}{n+2}$, the output of efficient firms is $q_e^B = \frac{1+c}{n+2}$ and profits of firms is $\pi_i^B = \left(q_i^B\right)^2 \; i = c, e$. Observe that $c < \frac{1}{n+1}$ for the inefficient firm to produce. Otherwise, the inefficient firm does not produce and the merger has no effect. After the merger, total output is given by $Q^A = \frac{n}{n+1}$, individual output by $q^A = \frac{Q^A}{n}$ and individual profits by $\pi^A = \left(q^A\right)^2$.

a) Gross consumer sursplus as a function of sales is given by $CS(Q) = Q - \frac{Q^2}{2}$. Social Welfare before the merger is lower than after the merger when:

$$\begin{aligned} CS(Q^A) &> CS(Q^B) - cq_e^B \\ \text{or } c &> \frac{2n+3}{2n^3+8n^2+9n+3} \end{aligned} \tag{14}$$

The market share of the inefficient firm as a function of c is:

$$s(c) = \frac{q_c^B}{Q^B} = \frac{1-(n+1)c}{n+1-c}$$

It decreases with c. Then when (14) holds

$$s(c) < s\left(\frac{2n+3}{2n^3+8n^2+9n+3}\right) = \frac{2(n+1)}{2n^2+6n+5}$$

b) The merger is profitable if the firms that merge obtain more (joint) profits after the merger i.e.:

$$\begin{aligned} \pi^A &> \pi_e^B + \pi_c^B \\ \text{or } c &> \frac{n^2-2}{n^3+3n^2+4n+2} \end{aligned} \tag{15}$$

The ratio between premerger output of the efficient and ineffcient firms is given as a function of c by:

$$f(c) = \frac{q_e^B}{q_c^B} = \frac{1+c}{1-c(n+1)}$$

As it is increasing in c, when (15) holds

$$f(c) > f\left(\frac{n^2-2}{n^3+3n^2+4n+2}\right) = \frac{n(n+2)}{2(n+1)}$$

2. If the price set by the upstream firm (the wholesale price) is w, downstream firms will sell $\frac{n(\alpha - w)}{n+1}$. Then the profits of the upstream firm as a function of w are:

$$\frac{n(\alpha - w)}{n+1}(w - c_u) \tag{16}$$

They are maximized at $w^* = \frac{\alpha + c_u}{2}$. Evaluating (16) at w^* we have:

$$\left(\frac{\alpha - c_u}{2}\right)^2 \left(\frac{n}{n+1}\right) \tag{17}$$

Given that $\left(\frac{\alpha - c_u}{2}\right)^2$ are the monopoly profits, (17) tends to the monopoly profits when n tends to infinity.

3. To derive the equilibrium contracts is good to think in the optimal contract between the upstream firm and one downstream firm i holding constant the output of the other firms to q_{-i}. In this case, Firm i will accept the contract if

$$F \leq \left(\frac{\alpha - q_{-i} - w_i}{2}\right)^2 - \left(\frac{\alpha - q_{-i} - c}{2}\right)^2$$

and will produce:

$$q_i = \frac{\alpha - q_{-i} - w_i}{2}$$

Then the upstream firm will set the contract to maximize:

$$\begin{aligned} \underset{w_i, F}{Max} (w_i - c_u) \left(\frac{\alpha - q_{-i} - w_i}{2}\right) + F \\ s.t.\ F \quad \leq \quad \left(\frac{\alpha - q_{-i} - w_i}{2}\right)^2 - \left(\frac{\alpha - q_{-i} - c}{2}\right)^2 \end{aligned}$$

Given that he optimally will choose the hightest fixed part that satisfies the inequality, the previous maximization programm can be written as:

$$\underset{w_i}{Max} (w_i - c_u) \left(\frac{\alpha - q_{-i} - w_i}{2}\right) + \left(\frac{\alpha - q_{-i} - w_i}{2}\right)^2 - \left(\frac{\alpha - q_{-i} - c}{2}\right)^2$$

The FOC is given by

$$\begin{aligned} \left(\frac{\alpha - q_{-i} - w_i}{2}\right) - (\frac{1}{2})(w_i - c_u) - (\frac{1}{2})(\alpha - q_{-i} - w_i) &= 0 \\ w_i &= c_u \end{aligned}$$

The fixed fee will be equal to:

$$F = \left(\frac{\alpha - q_{-i} - c_u}{2}\right)^2 - \left(\frac{\alpha - q_{-i} - c}{2}\right)^2$$

Then the downstream firm will obtain:

$$\left(\frac{\alpha - q_{-i} - c}{2}\right)^2 \tag{18}$$

As in equilibrium for all firms the wholesale price will be equal to marginal cost, they will sell the Cournot outputs with these costs:

$$q_c = \frac{\alpha - c_u}{n+1}$$

Then replacing in (18) q_{-i} by $(n-1)q_c$, we obtain the payoffs of downstream firms:

$$\left(\frac{2(\alpha - c) - (n-1)(c - c_u)}{2(n+1)}\right)^2$$

4. a) Denote franchises by subindex i, $i = 1...n$ In the fourth stage n independent franchisees, each one with marginal cost w_i, and Firm B, with cost c, compete in the market. Given the results in class the output sold in equilibrium by each franchise and its profits are given respectively by:

$$\begin{aligned} q_i &= \frac{\alpha - (n+1)w_i + c + \sum_{j\neq i} w_j}{n+2} \\ \pi_i &= q_i^2 \end{aligned}$$

In the third stage, franchisees will accept any supply contract whose F_i is lower than the profits they are going to get in the fourth stage. In the second stage, Firm A will set the fixed part exactly equal to the profits that each franchisee is going to obtain in the fourth stage. Then when setting the wholesale prices, Firm A will maximize:

$$\sum_{i=1}^{n}(w_i - c)\left(\frac{\alpha - (n+1)w_i + c + \sum_{j\neq i} w_j}{n+2}\right) + \left(\frac{\alpha - (n+1)w_i + c + \sum_{j\neq i} w_j}{n+2}\right)^2$$
$$\sum_{i=1}^{n}\left(\frac{\alpha - (n+1)w_i + c + \sum_{j\neq i} w_j}{n+2}\right)\left((w_i - c) + \left(\frac{\alpha - (n+1)w_i + c + \sum_{j\neq i} w_j}{n+2}\right)\right)$$
$$\left(\frac{\alpha + \sum_{i=1}^{n} w_i - (n+1)c}{n+2}\right)\sum_{i=1}^{n}\left(\frac{\alpha - (n+1)w_i + c + \sum_{j\neq i} w_j}{n+2}\right)$$

$$\left(\frac{\alpha + \sum_{i=1}^{n} w_i - (n+1)c}{n+2}\right)\left(\frac{n\alpha - 2\sum_{i=1}^{n} w_i + nc}{n+2}\right) \tag{19}$$

The maximization proces will not determine the individual values w_i but only its sum. The optimal sum amounts to:

$$\sum_{i=1}^{n} w_i = \frac{(n-2)\alpha + (3n+2)c}{4} \tag{20}$$

Plugging (20) into (19) we obtain the profits Firm A gets by setting the optimal contract:

$$\frac{(\alpha - c)^2}{8}$$

They amount to the profits the Stacklerberg leader will obtain in this market. The role of the observable contracts is to turn Firm A into a Stackelberg leader.

Then the objective in the First Stage is:

$$\frac{(\alpha - c)^2}{8} - n\varepsilon$$

It is decreasing in n. Therefore Firm A will set only 1 independent franchise. Setting franchises or contracts have the equivalent effect of allowing Firm A to commit to a higher output in the marketplace. Given that franchises are costly and contracts are free, Firm A chooses the latter option.

b) Firm B does not observe the supply contracts. Then, optimal supply contracts have to be calculated holding constant the output of firm B (q_B). In other words the optimal supply contracts have to be calculated as if Firm A served demand $P = (\alpha - q_B) - Q$. In class we solved this problem and I only recall the result. Wholesale prices will be set such that output of franchises (q_A) is equal to the monopoly output

$$q_A = \left(\frac{\alpha - q_B - c}{2} \right) \tag{21}$$

On the other hand, in equilibrium, Firm B should maximize profits given the output of franchises. This implies

$$q_B = \left(\frac{\alpha - q_A - c}{2} \right) \tag{22}$$

Solving (21) and (22), we obtain the equilibrium outputs:

$$q_A = q_B = \frac{\alpha - c}{3}$$

With the fixed fee, Firm A will extract all the rents from the franchises. He will obtain the profits generated by the sales of the franchises. They amount to

$$(\alpha - c - 2\left(\frac{\alpha - c}{3} \right)) \left(\frac{\alpha - c}{3} \right) = \left(\frac{\alpha - c}{3} \right)^2$$

Then, in the first stage, the profits of Firm A as a function of the number of franchises is:

$$\left(\frac{\alpha - c}{3} \right)^2 - n\varepsilon$$

It is decreasing in n. Therefore, Firm A will set only 1 independent franchise. Any possible strategic effect of creating competing franchises is offset by the subsequent choice of contracts that will induce franchises to choose the joint profit maximization output.

c) A supply contract $(w_i q + F_i)$ is only observable to the franchise i that receives it. Then, optimal supply contracts have to be calculated holding constant the output of firm B (q_B) and the output of the other franchises $(\sum_{j\neq i} q_j)$. In other words the optimal supply contracts have to be calculated as if Firm A served demand $P = (\alpha - q_B - \sum_{j\neq i} q_j) - Q$. In class we solved this problem and I only recall the result. The fixed part extracts all the profits the franchisee is going to obtain in the fourth stage and the wholesale price is set equal to marginal cost $w_i = c$. Given those contracts, in the fourth stage both the franchises and Firm B will compete at cost c. Then everyone will produce:

$$\frac{\alpha - c}{n+2}$$

With the fixed fee, Firm A will extract all the rents from the franchises. He will obtain the profits generated by the sales of the franchises. They amount to:

$$(\alpha - c - (n+1)\left(\frac{\alpha - c}{n+2}\right))n\left(\frac{\alpha - c}{n+2}\right) = n\left(\frac{\alpha - c}{n+2}\right)^2$$

Then in the first stage the objective of Firm B will be:

$$\Pi = n\left(\frac{\alpha - c}{n+2}\right)^2 - n\varepsilon$$

$$\frac{\partial \Pi}{\partial n} = (\alpha - c)^2\left(\frac{-n+2}{(n+2)^3}\right) - \varepsilon = 0 \tag{23}$$

If $\varepsilon = 0$, the solution to (23) is $n = 2$. This should be the optimal option for ε low enough. Then Firm A sets two franchises.

6.)

a) The objective of U when he sets a linear price of w and extracts all the rents through the fixed fee is given by:

$$\begin{aligned}
\pi_i &= \left(\frac{a - w}{2}\right)\left(\frac{2w + (1-d)(a-w)}{2}\right) \\
\frac{\partial \pi_i}{\partial w} &= -2w + 2d(a - w) = 0 \\
w^* &= \frac{da}{1+d}
\end{aligned}$$

It replicates the situation with vertical integration. The output with vertical integration is $\frac{a}{2(1+d)}$ and $\frac{a - w^*}{2} = \frac{a}{2(1+d)}$.

b)The objective of U is given by:

$$\begin{aligned}\pi_i &= \left(\frac{a-w}{2}\right)\left(\frac{2w-d(a-w)}{2}\right)\\ \frac{\partial \pi_i}{\partial w} &= -w(2+d)+da+(a-w)(2+d)=0\\ w^+ &= \frac{a(1+d)}{2+d}\end{aligned}$$

We have $w^* < w^+$, then welfare is higher with a two-part tariff contract.

7. Assume that the equilibrium we are looking for is given by:

$$(w_1^*, w_2^*, q_1^*(w_1), q_2^*(w_2)) \tag{24}$$

Observe that the strategy of downstream firms specify an output for any contract they may receive. To be an equilibrium every agent should be maximizing. As far as downstream firms are concerned, this implies that they maximize profits given the ouput chosen by the competitor in equilibrium.

$$q_1^*(w_1) = \frac{\alpha - q_2(w_2^*) - w_1}{2}$$

$$q_2^*(w_2) = \frac{\alpha - q_1(w_1^*) - w_2}{2}$$

For Firm U, it means that the contracts it sets maximize its revenues. As prices are not observed by competitors, when Firm U changes the price to Firm 1, Firm 2 still behaves as prescribed by the equilibrium and obtains the same revenues from it. Therefore, we have to check that Firm U maximizes the revenues he can separately extract from each downstream firm. If he sets w_1, Firm U will obtain:

$$\begin{aligned}I(w_1) &= w_1 q_1^*(w_1)\\ \frac{\partial I(w_1)}{\partial w_1} &= q_1^*(w_1) - \frac{1}{2}w_1 = 0\\ w_1 &= \frac{\alpha - q_2^*(w_2^*)}{2}\end{aligned}$$

Then for (24) to be an equilibrium we must have that:

$$w_1^* = \frac{\alpha - q_2^*(w_2^*)}{2} \tag{25}$$

- We have a symmetric condition for Firm 2:

$$w_2^* = \frac{\alpha - q_1^*(w_1^*)}{2} \tag{26}$$

From

$$q_1^*(w_1^*) = \frac{\alpha - q_2(w_2^*) - w_1^*}{2}$$

$$q_2^*(w_2^*) = \frac{\alpha - q_1(w_1^*) - w_2^*}{2}$$

we can obtain that

$$q_1^*(w_1^*) = \frac{\alpha - 2w_1^* + w_2^*}{3} \tag{27}$$

$$q_2^*(w_2^*) = \frac{\alpha - 2w_2^* + w_1^*}{3} \tag{28}$$

Plugging (27) and (28) into (25) and (26) and solving the equations we obtain that:

$$w_1^* = w_1^* = \frac{2}{5}\alpha$$

Then

$$q_1^*(w_1^*) = q_2^*(w_2^*) = \frac{\alpha}{5}$$

9 Problem Set V. Product Differentiation.

1. (Tirole (1988) p. 282) Consider the model of differentiation on the line. The two firms are located at the two extremities of the segment ($x = 0$ and $x = 1$). Transportation costs are linear in distance ($C(d) = td$) and the distribution of consumers is uniform along the segment. The firms have constant marginal costs, c_1 and c_2, which are not necessarily equal (but, for simplicity, assume that they do not differ too much, so that each firm has a positive market share in equilibrium).

Compute the Nash Equilibrium in prices.

2. (Eaton and Lipsey (1975) Review of Economic Studies; 42(1) p. 27-49). Assume that consumers are uniformly distributed in the segment of length 1. Each consumer wants to buy one and only one unit of a good. The price of the good is regulated at a level P greater than the unit cost. Firms decide where to locate (they only can choose one location). Find the equilibrium when there are 4 or 5 firms in the market. (With 3 firms there is no equilibrium).

3. Assume that consumers are uniformly distributed on a circle with a perimeter equal to 1. Consumers wish to buy one unit of the good and have linear transportation costs in distance ($C(d) = td$). Analyze the Subgame Perfect Nash Equilibrium of the following game.

In the first stage, potential firms decide whether to enter into the market or not. Firms that enter have to pay a fixed cost F.

Firms that have entered are automatically located equidistant from one another on the circle.

In the second stage, firms that have entered compete in prices. For all firms, marginal cost is constant and equal to c.

Compare the equilibrium number of firms with the one that would minimize the transportation costs.

4. (Cabral (1997) p.86-7) (BRAND PROLIFERATION) Assume that consumers are uniformly distributed in the segment of length 1. Each consumer wants to buy one and only one unit of a good. The price of the good is regulated

at a level P greater than the unit cost c. Two firms, A and B, can produce the good. Firms are allowed to locate in more than one location. The cost of each location is F. We have that:

$$\frac{P-c}{4} < F < \frac{P-c}{2}$$

Analize the Subgame Perfect Nash Equilibrium of the following game.

In the first stage, Firm A chooses its locations.

In the second stage, Firm B chooses its locations.

5. (Motta (1993)) Assume that utility of consumers is given by $\theta q - p$, where θ is a taste for quality, q is quality and p is price. θ is uniformly distributed in $[0,1]$. Consumers want to consume either one or zero units of the differentiated good. No consumption gives an utility of zero. Firm 2 produces a good of quality $q_2 = 1$. Firm 1 produces a good of quality q_1, where $0 < q_1 < q_2 = 1$. Production costs are zero. Find the equilibrium in prices. Observe that there will be consumers that will prefer not to buy the good i.e. the market will not be covered. Check that contrary to what happened when the market was covered the profit of the low-quality firm is not always decreasing in its own quality.

6. Suppose that the price of an industry is regulated at a level $p > c$ where c is marginal cost. Firm i can spend z_i to improve its market share

$$\frac{z_i^\phi}{\sum_{j=1}^n z_j^\phi} \text{ where } 0 < \phi \leq 2.$$

The profit of firm i is thus

$$\pi_i = (p-c)S\frac{z_i^\phi}{\sum_{j=1}^n z_j^\phi} - z_i - F$$

where S is the size of the market and F the fixed cost of entry. Compute the (symmetric) free entry equilibrium (the SPE of the two-stage entry-choice of z_i game). Study the asymptotics of the equilibrium when F tends to 0.

7. Consumers are uniformly distributed in the segment $[0,1]$. Each consumer wants to buy one unit of the good. Firm A can locate in any point in the segment $[-\frac{1}{2}, 0]$, whereas Firm B can locate in any point in $[1, \frac{3}{2}]$. The unit cost of producing the good is c. Solve the following two-stage game: In the first stage, firms decide whether to locate. In the second stage, they compete in prices. Explain the result and compare it with other models in the literature.

(Hint: the second stage equilibrium, given that Firm A locates in a and Firm B locates in b is given by:

$$\begin{aligned} p_A^* &= c + \frac{(2+a+b)t}{3} \\ p_B^* &= c + \frac{(4-a-b)t}{3} \end{aligned}$$

)

10 Solution to Problem Set V. Product Differentiation.

1.(See Tirole (1993)p. 301)

$$p_i = t + \frac{2c_i + c_j}{3}$$

2. With 4 firms two locate in 1/4 and the other two in 3/4.

With 5 firms, two locate in 1/6, one in 1/2 and the rest in 5/6.

3. (See Tirole (1993) pp. 282-4) In the second stage profits given that n firms have entered are: $\frac{t}{n^2}$.

From the zero profit condition the number of firms is obtained:

$$\begin{aligned} \frac{t}{n^2} &= F \\ n^c &= \sqrt{\frac{t}{F}} \end{aligned}$$

The number of firms that minimizes transportation cost is:

$$n^* = \frac{1}{2}\sqrt{\frac{t}{F}} < n^c$$

4. If firm A chooses only one location, it will locate it in $\frac{F}{P-c}$(or $1-\frac{F}{P-c}$). Then Firm B will locate in $\frac{F}{P-c}+\varepsilon$ and Firm A will obtain zero profits. In all other locations obtains negative profits. If it locates in $x \in [0, \frac{F}{P-c})$, Firm B will locate in $x+\varepsilon$ and Firm A will obtain $(P-c)x - F < 0$. If Firm A locates in $x \in (\frac{F}{P-c}, 1-\frac{F}{P-c})$, Firm B will locate both in $x-\varepsilon$ and $x+\varepsilon$ and Firm A will obtain $-F$.

If Firm A chooses two locations, it will locate them in 1/4 and 3/4 to reduce the demand that Firm B can capture. Then Firm B will decide not to enter, because it can only obtain the fourth of total demand and $\frac{P-c}{4} < F$. Then Firm A will obtain $P - c - 2F > 0$. Therefore, firm A will choose two locations.

5. The indifferent consumer between buying the good of low and high quality is:

$$\theta_1 = \frac{p_2 - p_1}{1 - q_1}$$

The indifferent consumer between buying the good of low quality and not buying is:

$$\theta_0 = \frac{p_1}{q_1}$$

Then the profits of the low (firm 1) and high (firm 2) quality firms are given respectively by:

$$\pi_2 = (1-\theta_1)p_2 \text{ and } \pi_1 = (\theta_1 - \theta_0)p_1$$

Equilibrium prices are given by:

$$p_1^* = \frac{q_1(1-q_1)}{4-q_1} \text{ and } p_2^* = \frac{2(1-q_1)}{4-q_1}$$

The profits in equilibrium are:

$$\pi_1^* = \frac{q_1(1-q_1)}{(4-q_1)^2} \text{ and } \pi_2^* = \frac{4(1-q_1)}{(4-q_1)^2}$$

Then,

$$sign[\frac{\partial \pi_1^*}{\partial q_1}] = sign[-7q_1 + 4]$$

Contrary to what happened when the market was covered, it is not always optimal for the low quality firm to deteriorate its quality in order to reduce the competition with the high quality firm. Now this has the cost of inducing some consumers not to buy the good.

6. Given that n firms have entered the market, the profit of one of them is given by:

$$\pi_i = (P-c)A\frac{z_i^\phi}{\sum_{j=1}^n z_j^\phi} - z_i$$

The FOC is given by:

$$(P-c)A\frac{\phi z_i^{\phi-1}\left(\sum_{j=1}^n z_j^\phi\right) - \phi z_i^{2\phi-1}}{\left(\sum_{j=1}^n z_j^\phi\right)^2} - 1 = 0$$

To obtain the symmetric equilibrium we impose it in the FOC:

$$(P-c)A\phi\left(\frac{(n-1)z^{2\phi-1}}{n^2 z^{2\phi}}\right) - 1 = 0$$

The equilibrium level of investment (to improve market share) is:

$$z^* = (P-c)A\phi\left(\frac{n-1}{n^2}\right)$$

The zero profit condition is given by:

$$(P-c)\frac{A}{n} - (P-c)A\phi\left(\frac{n-1}{n^2}\right) - F = 0 \qquad (29)$$

If we let $B = \frac{F}{(P-c)A}$, (29) can be rewritten as:

$$(P-c)A\left(\frac{1}{n} - \phi\left(\frac{n-1}{n^2}\right) - B\right) = 0$$

The number of firms in the free entry equilibrium is given:

$$n^* = \frac{1-\phi+\sqrt{(1-\phi)^2+4B\phi}}{2B}$$

When $A->\infty$, $B->0$. Then, if $\phi<1$, we have

$$\underset{B->0}{Lim} n^* = \frac{2(1-\phi)}{0} = \infty$$

If $\phi=1$, $n^* = \frac{1}{\sqrt{B}}$. Then

$$\underset{B->0}{Lim} n^* = \infty$$

If $1<\phi\leq 2$, then

$$\underset{B->0}{Lim} n^* = \frac{1-\phi+\phi-1}{0} = \text{ Indeterminate}$$

Observe that the value of the square root when $B->0$ is $\phi-1$, because we have to take its positive solution.

We apply L'Hôpital rule and then

$$\underset{B->0}{Lim}(1/2)\frac{4\phi}{2\sqrt{(1-\phi)^2+4B\phi}} = \frac{\phi}{\phi-1}$$

7. Demand given prices is obtained by identifying the indifferent consumer:

$$\begin{aligned} p_A+t(x-a) &= p_B+t(b-x) \\ x(p_A,p_B) &= \frac{p_B-p_A+t(b+a)}{2t} \end{aligned}$$

With prices and demand we can obtain the profits of firms as a function of locations. Profits of Firm A and Firm B are given respectively by:

$$\begin{aligned} \Pi_A &= (p_A^*-c)x(p_A^*,p_B^*) = \frac{1}{18}(2+a+b)^2 t \\ \Pi_B &= (p_B^*-c)(1-x(p_A^*,p_B^*)) = \frac{1}{18}(4-a-b)^2 t \end{aligned}$$

As we have that $\frac{\partial \Pi_A}{\partial a}>0$ and $\frac{\partial \Pi_B}{\partial b}<0$, the first stage equilibrium will be $a^*=0$ and $b^*=1$.

11 Problem Set VI: Repeated Interaction.

1. Assume that market demand is given by $X = S(a - P)$. Firms have unit cost c and compete à la Cournot. This market game is repeated infinite times. Find the values of the discount factor such that the symmetric monopoly solution can be sustained in equilibrium using trigger strategies (reversion to the static Nash equilibrium).

2. Assume that everything is like in question 1 except that there are only two firms. For low values of the discount factor the monopoly solution can not be sustained in equilibrium. In those cases, find the (symmetric) subgame-perfect equilibrium that yields the firms the greatest profits, called the extremal equilibrium.

3. (Bernheim and Whinston (1990)) Assume that we have two firms that compete à la Bertrand, demand is given by $D(p)$ and the cost function is given by: $C(q) = cq + F$ if $q > 0$ and 0 otherwise. This market game is repeated infinite times. Obtain the discount factor that sustain the collusive outcome: both firms set the monopoly price and share equally demand. Assume that firms use "trigger strategies". (In the static equilibrium, assume that although both firms set the same price, all demand goes to one firm).

4. (Deneckere (1983)) Assume we have two firms (firm 1 and firm 2) that compete à la Cournot. Firm i ($i = 1, 2$) produces good i whose demand expressing the price of their product, p_i, in terms of the production levels q_i is given by:

$$\begin{aligned} p_i &= 1 - q_i - \gamma q_j \\ i, j &= 1, 2 \; i \neq j \text{ and } 1 > \gamma > 0 \end{aligned}$$

There are no production costs. This market game is repeated infinite times. Find the values of the discount factor such that the monopoly solution can be sustained in equilibrium using trigger strategies (reversion to the static Nash equilibrium). Study the effect of product differentiation on the possibility of collusion.

12 Solution to Problem Set VI: Repeated Interaction.

1. In class we derived that the discount factor should satisfy:

$$\delta \geq \frac{\bar{\pi}_i - \pi_i^*}{\bar{\pi}_i - \hat{\pi}_i} \tag{30}$$

$\bar{\pi}_i$represents the one-period deviation profits, π_i^* the (symmetric) collusive profits and $\hat{\pi}_i$ the profits in the equilibrium of the one-shot game.

In this case, we have

$$\pi_i^* = \frac{S}{n}\left(\frac{a-c}{2}\right)^2 \tag{31}$$

$$\hat{\pi}_i = S\left(\frac{a-c}{n+1}\right)^2 \tag{32}$$

One period profits of a firm if he deviates and produces q_i and competitors stick to the agreement are given by:

$$\pi_i = (a - (n-1)\left(\frac{S}{n}\left(\frac{a-c}{2}\right)\right) - q_i)q_i$$

Then

$$\bar{\pi}_i = S\left(\frac{(a-c)(n+1)}{4n}\right)^2 \tag{33}$$

Plugging (31),(32) and (33) into (30) we have:

$$\delta \geq \left(1 + \frac{4n}{(n+1)^2}\right)^{-1}$$

2. For $\delta < \frac{9}{17}$, the monopoly symmetric solution can not be sustained. In those cases the best symmetric equilibrium will be the lowest output that satisfies the incentive compatibility constraint. Assume that everybody plays q, then each firm obtains $(a-c-\frac{2q}{S})q$. If a firm deviates, it obtains $\frac{(S(a-c)-q)^2}{4S}$. Then firms want to produce q (no to deviate) if:

$$\left(\frac{1}{1-\delta}\right)(a-c-\frac{2q}{S})q - \frac{(S(a-c)-q)^2}{4S} - \left(\frac{\delta}{1-\delta}\right)\left(\frac{a-c}{3}\right)^2 \geq 0$$

The lowest output that satisfies this equation is:

$$q^* = \frac{S(a-c)(9-5\delta)}{3(9-\delta)}$$

It is greater than half the monopoly output if $\delta < \frac{9}{17}$. It is decreasing in δ and $(a-c-\frac{2q^*}{S})q^*$ is increasing in δ if $\delta < \frac{9}{17}$.

3. In the one-shot equilibrium both firms set p_c such that

$$(p_c - c)D(p_c) = F$$

and all demand goes to one firm. (Observe that in this case reversion to the one-shot Nash equilibrium is an optimal punishment because it yields a payoff of zero). Then the monopoly solution where every firm obtains: $\frac{\pi^m}{2} - F$, where

$$\pi^m = \max_p (p-c)D(p)$$

can be sustained if:

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{\pi^m}{2}-F\right) \geq \frac{\pi^m}{2}$$
$$\delta \geq \frac{\pi^m}{2(\pi^m - F)}$$

If $F = 0$, we obtain the well-known result. The greater the fixed costs, the harder to sustain collusion.

4. First of all, we derive the profits in the one-shot Nash equilibrium (π^N), in the joint profit maximization solution (monopoly) and the deviation profits from the monopoly solution.

One-shot Nash equilibrium:

Profits of firm i are given by:

$$\pi_i = (1 - q_i - \gamma q_j)q_i$$

$$\frac{\partial \pi_i}{\partial q_i} = 1 - 2q_i - \gamma q_j = 0 \quad (34)$$
$$i, j = 1, 2 \; i \neq j \quad (35)$$

Solving the system we obtain the (symmteric) equilibrium $q^N = \dfrac{1}{2+\gamma}$. Then profits are given by $\pi^N = \left(\dfrac{1}{2+\gamma}\right)^2$.

Joint-profit maximization:

Joint profits are given by:

$$\Pi = (1 - q_1 - \gamma q_2)q_1 + (1 - q_2 - \gamma q_1)q_2$$

$$\frac{\partial \Pi}{\partial q_i} = 1 - 2q_i - 2\gamma q_j = 0$$
$$i, j = 1, 2 \; i \neq j$$

Solving the system we obtain the (symmteric) joint-profit maximization outputs $q^* = \dfrac{1}{2(1+\gamma)}$. Then profits (per firm) are given by $\pi^* = \dfrac{1}{4(1+\gamma)}$.

Deviation profits:

If firm j produces q^*, then the profits of firm i are given by:

$$\pi_i = (1 - q_i - \frac{\gamma}{2(1+\gamma)})q_i$$

$$\frac{\partial \pi_i}{\partial q_i} = 1 - \frac{\gamma}{2(1+\gamma)} - 2q_i = 0$$
$$q_i^D = \frac{2+\gamma}{4(1+\gamma)}$$

Then, deviation profits are given by: $\pi^D = \left(\frac{2+\gamma}{4(1+\gamma)}\right)^2$.

Producing the joint-profit maximization outputs can be sustained as a Subgame Perfect Nash Equilibrium of the infinitely repeated game if the following condition holds:

$$\frac{\pi^*}{1-\delta} \geq \pi^D + \frac{\delta \pi^N}{1-\delta}$$

$$\delta \geq \underline{\delta} = \frac{\pi^D - \pi^*}{\pi^D - \pi^N} \tag{36}$$

As we have:

$$\begin{aligned} \pi^D - \pi^* &= \frac{\gamma^2}{(4(1+\gamma))^2} \\ \pi^D - \pi^N &= \frac{\gamma^2(8+8\gamma+\gamma^2)}{(4(1+\gamma))^2(2+\gamma)^2} \end{aligned}$$

Then

$$\underline{\delta} = \frac{(2+\gamma)^2}{8+8\gamma+\gamma^2}$$

We have that $\frac{\partial \underline{\delta}}{\partial \gamma} > 0$. Then when goods become more differentiated (lower γ) the set of values of the discount factor such that (36) holds becomes larger. Then, product differentiation helps collusion to take place.