1 Competition in quantities.

1.1 The linear model.

Assume we have a market whose inverse demand is given by:

$$P(Q) = a - bQ$$

n firms, denoted with a natural number from 1 to n, operate in this market. Firms have constant marginal costs (not necessarily equal) denoted by c_i ($i = 1...n$). Firms choose output (i.e. they compete à la Cournot) q_i and market price is such that demand equals the quantity supplied by firms:

$$P = a - b\sum_{i=1}^{n} q_i$$

This allows us to write the profits of firms as a function of the quantity they supply:

$$\pi_i = (a - b\sum_{i=1}^{n} q_i - c_i)q_i$$

We proceed to find the Nash (Cournot) equilibrium of this game. A vector of outputs is an equilibrium if each firm maximizes its profits. Given that second order conditions are satisfied, this is equivalent to find the solution to the n-equation system conformed by the following FOC:

$$\begin{aligned} \frac{\partial \pi_i}{\partial q_i} &= a - b\sum_{j=1}^{n} q_j - c_i - bq_i = 0 \\ i &= 1...n \end{aligned}$$

To solve the system we have:

$$\sum_{i=1}^{n} \frac{\partial \pi_i}{\partial q_i} = n(a - bQ) - \sum_{i=1}^{n} c_i - bQ = 0$$

The total output in equilibrium is given by:

$$\begin{aligned} Q &= \frac{n(a - \bar{c})}{b(n+1)} \\ \text{where } \bar{c} &= \frac{\sum_{i=1}^{n} c_i}{n} \end{aligned}$$

Price, individual outputs and profits are given respectively by:

$$\begin{aligned} P &= \frac{a + n\bar{c}}{n+1} \\ q_i &= \frac{a - nc_i + \sum_{j \neq i} c_j}{b(n+1)} \\ \pi_i &= bq_i^2 \end{aligned}$$

Social Welfare (sum of consumer surplus and profits) in this market amounts to:

$$W = CS + \sum_{i=1}^{n} \pi_i = \int_0^Q (a - bx)dx - \sum_{i=1}^{n} c_i q_i$$

It is possible to check that the "intuitive" comparative statics hold in this model. In the following Section we check that it depends on a general condition (1) defined below. In particular we have

$$\frac{\partial Q}{\partial c_i} < 0,\ \frac{\partial q_i}{\partial c_i} < 0,\ \frac{\partial q_j}{\partial c_i} > 0,\ \frac{\partial \pi_i}{\partial c_i} < 0,\ \frac{\partial \pi_j}{\partial c_i} > 0$$

Given that the increase in the cost of a firm reduces consumer surplus (because it reduces output) but it increases the profits of competitors, it has an ambiguous effect on welfare.

To analyze the effect on entry we focus in the symmetric case. Then we have the following:

$$\frac{\partial Q}{\partial n} > 0,\ \frac{\partial q_i}{\partial n} < 0,\ \frac{\partial \pi_i}{\partial n} < 0$$

Before presenting the general model, we will allow for a graphical representation of the Cournot model. We are going to consider the duopoly case (with firm 1 and firm 2).

We represent graphically the equilibrium by plotting the F.O.C. of each firm. From the F.OC. of firm 1, we can obtain the output that maximizes the profits of firm 1 as a function of the output of firm 2. This function is called the best response of firm 1. We can plot in the plane (q_1, q_2)

$$\begin{aligned} \frac{\partial \pi_1}{\partial q_1} &= a - b(q_1 + q_2) - c_1 - bq_1 = 0 \\ q_1 &= \frac{a - c_1 - bq_2}{2b} \end{aligned}$$

If firm 2 does not produce the optimal output of firm 1 is $\frac{a - c_1}{2b}$ that is its monopoly output. The optimal output of firm 1 is 0 when $q_2 = \frac{a - c_1}{b}$. With these two pieces of information, we can plot the best response of firm 1 because it is a straight line.

We can proceed likewise with firm 2.

The equilibrium is obtained in the crossing point of both best-responses. One important thing to notice is that best responses are downward sloping. This is the "natural" thing for Cournot competition. See exercises for an exception.

In the next Section we present a more general model.

1.2 Existence and unicity.

Consider a Cournot game where demand is given by a function twice continuously differentiable $P(Q)$. We have that $P' < 0$, and it also satisfies:

$$P' + P"Q < 0 \tag{1}$$

Costs of firms ($C_i(q_i)$) are strictly increasing and

$$C_i" - P' > 0. \tag{2}$$

Furthermore, there exists a quantity $\bar{Q}$, such that if $Q \geq \bar{Q}$ we have that $P(Q) = 0$.

The FOC of a firm is given by:

$$\frac{\partial \pi_i}{\partial q_i} = P(Q) + q_i P'(Q) - C_i'(q_i) = 0 \tag{3}$$

This equation defines the best-response function of firm i, denoted by $r_i(q_{-i})$. It gives the profit-maximizing output of firm i as a function of the output of competitors. The SOC are satisfied, because the profit function is strictly concave:

$$\frac{\partial^2 \pi_i}{(\partial q_i)^2} = [P'(Q) + q_i P"(Q)] + [P'(Q) - C_i"(q_i)] < 0$$

The crucial point in the proof of existence and unicity is to check that the reaction function ($r_i(q_{-i})$) has a negative slope greater than -1. Applying the implicit function theorem to (3) we obtain that:

$$r_i'(q_{-i}) = -\frac{P'(Q) + q_i P"(Q)}{2P'(Q) + q_i P"(Q) - C_i"(q_i)}$$

We have just checked that both the numerator and the denominator are negative, but given (2), the denominator is greater in absolute value than the numerator. Therefore, $-1 < r_i'(q_{-i}) < 0$.

We define the cumulative best response $\phi_i(Q)$ as the output of firm i that maximizes its profits given that total output is Q. It is the unique solution to:

$$\begin{aligned} q_i &= r_i(Q - q_i) \\ q_i - r_i(Q - q_i) &= 0 \\ \frac{\partial q_i}{\partial Q} &= \phi_i'(Q) = \frac{r_i'(q_{-i})}{1 + r_i'(q_{-i})} < 0 \end{aligned}$$

$\phi_i(Q)$ is well-defined for $Q \geq r_i(0)$.

To illustrate what we have done algebraically, we can plot the best response of firm i as a function of the output of the other firms q_{-i}. We know that it is downward sloping with slope greater than -1. We can plot the function:

$$q_i + q_{-i} = Q$$

The combination of points that lead to the same aggregate output Q. It is a line with slope -1. The crossing point of both lines gives $\phi_i(Q)$ the output of firm i that maximizes its profit when total output is Q. Then graphically we can observe that it is downward sloping. When Q increases the line representing the same level of aggregate output moves to the right and $\phi_i(Q)$ diminishes. When

$Q = r_i(0)$, $\phi_i(r_i(0)) = r_i(0)$. And consequently when $Q < r_i(0)$ the function $\phi_i(Q)$ is not well-defined because both lines do not cross in the region where outputs are positive.

The total output in equilibrium is given by the fixed point of $\Phi(Q) = \sum_{i=1}^{n} \phi_i(Q)$. It is well-defined for $[\max_i r_i(0), \bar{Q}]$. We are going to check that $\Phi(Q)$ crosses once and only once with the 45^o degree line. This will prove existence and unicity. It is downward sloping because it is the sum of downward sloping functions. Furthermore, we have that $\Phi(\bar{Q}) = 0$, because if total output is $\bar{Q}$ (or greater) price is zero and no firms wants to produce. The last thing we have to prove is that

$$\Phi(\max_i r_i(0) \geq \max_i r_i(0)).$$

To prove it, assume, without loss of generality, that

$$\max_i r_i(0) = r_j(0)$$

and recall that

$$\phi_j(r_j(0)) = r_j(0)$$

The line representing total output of $r_j(0)$ crosses the best response of firm j precisely in $r_j(0)$.

Then,

$$\Phi(\max_i r_i(0)) = \Phi(r_j(0)) = \phi_j(r_j(0)) + \sum_{i \neq j} \phi_i(r_j(0)) \geq r_j(0) = \max_i r_i(0)).$$

1.3 Comparative statics.

We are going to check that the same results on comparative statics obtained in the linear model hold in this more general model. The key point is that cumulative best responses are strictly decreasing. Recall from the proof above that total ouput in equilibrium (with n firms) satisfies:

$$Q = \sum_{i=1}^{n} \phi_i(Q)$$

1.3.1 Entry.

Assume we have an indutry with n firms (called incumbents). Suddenly a new firm enters in the market. How does the equilibrium change? Assume that the entrant produces a positive output $y > 0$ (if the entrant does not produce nothing changes). Use superscript $'$ and $''$ to denote respectively pre and postentry variables.We want to study the evolution of total and individual outputs in equilibrium.

Assume that pre-entry total output in equilibrium is higher than post-entry $Q' \geq Q$". Equilibrium conditions require that

$$\begin{aligned} Q' &= \sum_{i=1}^{n} \phi_i(Q') \\ Q'' &= \sum_{i=1}^{n} \phi_i(Q'') + y \end{aligned}$$

Given that $\phi_i'(Q)$ <0, we have that

$$Q' = \sum_{i=1}^{n} \phi_i(Q') < \sum_{i=1}^{n} \phi_i(Q'') + y = Q''$$

But this is a contradiction with the initial assumption.

Then, we must have that $Q' < Q$", which implies that $q_i' = \phi_i(Q') > \phi_i(Q'') = q_i''$. With entry, total output increases, but individual output of incumbents firms decrease. As far as individual firms are concerned, we have that as the competitors of an incumbent firm produce more than before entry ($Q' - q_i' < Q'' - q_i''$), the profits of incumbents decrease with entry.

1.3.2 Changes in the cost function.

Assume that everything is as before except that the cost function of firm j is given by $\theta_j C_j(q_j)$. It is satisfied that

$$\theta_j C_j''(q_j) - P' > 0$$

The FOC of firm j is given by

$$P(Q) + q_j P'(Q) - C_j'(q_j) = 0$$

It defines the optimal output of firm i as a function of Q and θ_j: $\phi_j(Q, \theta_j)$. We have just proved that $\dfrac{\partial \phi_j}{\partial Q} < 0$ and that $\dfrac{\partial \phi_j}{\partial \theta_j} = \dfrac{C_j'(q_j)}{P'(Q) - \theta_i C_j''(q_j)} < 0$.

We are going to see how the equilibrium changes when θ_j increases from θ_j' to $\theta_j^{''}$.

Denote respectively by $'$ and $''$ the variables before and after the change. Assume that total output in equilibrium is higher after the change $Q'' \geq Q'$. Equilibrium conditions require:

$$\begin{aligned} Q' &= \sum_{i \neq j} \phi_i(Q') + \phi_j(Q', \theta_j') \\ Q'' &= \sum_{i \neq j} \phi_i(Q'') + \phi_j(Q'', \theta_j'') \end{aligned}$$

Given that $\sum_{i \neq j} \phi_i (Q') \geq \sum_{i \neq j} \phi_i (Q'')$ and $\phi_j(Q', \theta_j') > \phi_j(Q^{''}, \theta_j^{''})$, this would imply that $Q' > Q''$ what is a contradiction with the initial assumption. Then we have that $Q" < Q'$. This implies that $q_i" > q_i'$ and, therefore, $q_j" = Q" - \sum_{i \neq j} q_i" < Q' - \sum_{i \neq j} q_i' = q_j'$. Both total output and the output of the firm whose cost has increased decrease. The output of the other firms increase. Profits of j decrease because $\sum_i q_i" > \sum_i q_i'$ and the profits of the other firms increase because $Q' - q_i' > Q^{"} - q_i^{"}$

1.4 Market power.

The Cournot model is our first model with competition. We want to compare the outcome with the one obtained with perfect competition. We focus on the difference between price and marginal cost through calculating what is called the Lerner index for each firm:

$$L_i = \frac{P - C_i'}{P}$$

Manipulating the First Order Condition we obtain:

$$L_i = \frac{P - C_i'}{P} = \frac{s_i}{\eta} \text{ where } \eta = -\frac{P}{QP'} \text{ is the elasticity of demand.}$$

The market power of firms is increasing in their market shares. At the industry level, we can calculate the aggregate Lerner Index. It is obtained by weighting the individual Lerner indices by market shares and aggregating over firms:

$$\sum_{i=1}^{n} s_i L_i = \frac{H}{\eta} \text{ where } H = \sum_{i=1}^{n} s_i^2 \text{ is the Herfindahl index of concentration.}$$

The Cournot model, therefore, gives support to the idea that concentrated markets will have more important departures from marginal cost. It is worth noting that the Herfindahl index increases both because the number of active firms decrease and because firms become more asymmetric. The effect of asymmetries can be easily understood by rewriting the Herfindahl Index as the weigted sum of market shares where the weights are also the market shares:

$$H = \sum_{i=1}^{n} s_i s_i$$

Asymmetries matter because larger firms carry more weight. Then, given a number of firms, the Herfindahl Index is minimized when all firms have the same market shares and is maximized when all the output is concentrated in only one firm.

For the case of symmetric firms we have that the Lerner Index can be restated as (given that $H = \frac{1}{n}$):

$$\frac{P - C_i'}{P} = \frac{1}{n\eta}$$

Market power decreases when n increases and vanishes when n tends to infinity. In other words, the competitive outcome obtains as the limit case when n tends to infinity.

1.5 Model of free entry.

Once we have studied the comparative statics when the number of firms in a market changes, we can analyze a two-stage game where in the first stage (symmetric) firms decide whether to enter in the market by paying a fixed costs F and, in the second stage, firms compete à la Cournot. To ease the exposition we will consider the number of firms to be a continuous variable.

In the second stage, if n firms have entered the market, we have a game as the one we have studied so far. Cournot profits and outputs are given respectively by π_n and q_n. In the first stage, the equilibrium condition is given by the zero profit condition.

$$\pi_n = F \tag{4}$$

If F is smaller than the monopoly output, it has a unique solution because π_n is decreasing and profits tend to zero when n tends to infinity. Call n^* the solution to (4).

Assume now that a planner can decide the number of firms in the market but he can not affect the way firms compete in the market. In other words, the planner can affect the structure but not the conduct of a market. How would compare the optimal number of firms with the number of firms with free entry?

Vives (1999) unveils the trade-off involved in the comparison.

"Two forces related to the external effect provoked by entry are at work. On the one hand, firms do not capture the entire consumer surplus generated by their production because they can not price discriminate. This means that firms will be too cautious when entering. On the other hand, there is typically a business stealing effect: when a new firm enters the output of each incumbent diminishes. This fact, with prices above marginal costs, provides a tendency towards excess entry"

Let us draw the possible discrepancy between private and social incentives for the case of constant marginal costs. Entry increases ouput, but the output of the entrant is higher than the increase in total output, because incumbents reduce their output. In other words, part of the output of the entrant is obtained at the expense of incumbents. Profits of the entrant are easily identified graphically. The gains of society also are easily obtained. Part of the social gains are not included in the profits because as the price is uniform, firms can not capture all the consumer surplus. (Recall what happens in the monopoly case). The profits obtained by the entrants because it replaces part of the output of incumbents does not belong to the social gains of entry. Society gains only with the new output induced by entry. So we have defined two different effects of different sign: on the one hand, firms will have too few incentives to enter because they can not capture all the consumer surplus and on the other hand they will have too many incentives to enter because part of the profits are obtaind from stealing to incumbents. This last effect is called the "business-stealing effect" and it will be the one that dominates.

The optimal number of firms (n^S) maximizes:

$$W(n) = \int_0^{nq_n} P(Q)dQ - nC(q_n) - nF$$

The FOC is given by:

$$\begin{aligned} W'(n) &= P(nq_n)\left(q_n + \frac{n\partial q_n}{\partial n}\right) - C(q_n) - nC'(q_n)\frac{\partial q_n}{\partial n} - F = \\ & P(nq_n)q_n - C(q_n) - F + (P(nq_n) - C'(q_n))\,\frac{n\partial q_n}{\partial n} \end{aligned}$$

$$W'(n) = \pi_n - F + (P(nq_n) - C'(q_n))\frac{n\partial q_n}{\partial n} \tag{5}$$

As in Cournot, firms always have market power and entry reduces individual outputs, (**??**) implies:

$$W'(n) < \pi_n - F$$

Then for $n \geq n^*$ $W'(n) < 0$, what implies that

$$n^S < n^*$$

In other words, too many firms enter with free entry. We check that market power has not only negative allocative effects in the short term, but it also distorts the investment decisions in the long run. Without market power (4) and (5) would coincide.

If the number of firms has to be an integer, the result on excessive entry reads as:

$$n^S - 1 \leq n^*$$

If there is too little entry this is at most by one firm. It is easy to find an example with little entry. It may be the case that it is optimal that a firm enters in the market, whereas entry is not profitable because monopoly profits are lower than the fixed cost of entry. The social benefits of entry are greater than the private benefits because the firm can not capture the entire consumer surplus generated and there is no business-stealing effect because we are considering the entry of the first firm.

Before starting the second chapter, we are going to introduce the concept of residual demand. The demand that a firm can serve under competition (called residual demand) is determined by market demand and the strategies played by competitors. Let us illustrate the concept with a Cournot duopoly example. Market demand is given by:

$$Q = a - P$$

There are 2 firms (1 and 2) whose outputs are denoted by q_i $(i = 1, 2)$. Then if we take as given q_2, the demand that is left to firm 1 is:

$$Q = a - P - q_2$$

because at any price, q_2 units will be sold by firm 2. We can illustrate graphically how do we transform market demand into the residual demand of firm 1 given the output of firm 2. The monopoly output on the residual demand is exactly the best-reponse of firm 1 given q_2.

So once the residual demand is known, a firm is indifferent between choosing price or quantity. However, the crucial effect of the chosen strategy comes through the effect it has on the residual demand of competitors.

2 Price competition.

2.1 Bertrand competition.

In the Cournot case, firms decide the quantity they sell and the demand decides the selling price through the market clearing condition. In the Bertrand case, we have the opposite assumption: firms choose the price they want to set and demand decides the quantity they sell, because firms have to sell all the forthcoming demand at the price they set. In this case, demands of firms are easy to obtain. Only firms setting the lowest price in the market have a positive demand. The remaining firms have no demand at all. There are different assumptions on the way to share the demand among the firms that set the lowest price. The most natural one is that they share demand equally i.e. all firms that set the lowest price sell the same amount of the good. However, sometimes for an equilibrium to exist some alternative assumption has to be used e.g. that all demand is satisfied by only one of the firms chosen randomly that set the lowest price.

Before obtaining the equilibrium for different assumptions on costs, we draw the residual demand of a firm taken as given the price of its competitor.

Case I: Constant marginal cost and the same for all firms. In this case, a price vector is a Bertrand equilibrium if and only if at least two firms set the price equal to marginal cost and the other firms set higher prices . Then in equilibrium market price equals marginal cost and the competitive result is obtained independently of the number of firms. This surprising result is usually called the Bertrand paradox. The reason for the result is straightforward. If the lowest price in the market was greater than marginal cost, there would always be a firm, that is not serving all the demand at present, that would increase its profits by setting a price slightly lower than market price in order to increase its sales. It is a particluar equilibrium, because firms use weakly dominated strategies.

It will also be interesting to analyze the asymmetric case. If there are at least two firms with the lowest cost c, then the result with symmetric cost holds. If there is only one firm (called it A) with the lowest cost, existence of equilibrium requires that the sharing rule assigns all the demand to firm A when it sets the lowest price. Let c_1 be the second lowest marginal cost, assumed to be lower than the monopoly price of firm A. Then at least two firms (including A) setting the lowest price p is a Bertrand equilibrium if $p \in [c, c_1]$. It is easy to see that there is no profitable deviations.

Case II: Constant marginal cost and an avoidable fixed cost. $C(q) = F + cq$ if $q > 0$ and $C(0) = 0$. We assume that the sharing rule is such that in case of a price tie a firm is selected (randomly) to serve the whole market (if demand is shared equally there is no equilibrium). Then the unique Bertrand equilibrium is for all firms to name the least break even monopoly price $\overline{p}$ (That is the least price that equates the monopoly profit to zero $(p - c)D(p) - F = 0$ PICTURE 0). Indeed a firm that undercuts this price would make negative profits and a higher price entails no sales.

It is worth noting that a very different outcome obtains if we analyze the two-stage entry game analyzed in the previous chapter. In the first stage, firms decide whether to pay the fixed costs and enter the market, and in the second stage firms that have entered the market compete in prices. If more than one firm enters in the second stage firms obtain zero profits because price equals marginal cost. Then the only way to recover the fixed cost is that only one firm enters the market and this is the equilibrium of the game. Then price equals the monopoly price and it is much higher than in the previous case. One can also compare the situation with the one with Cournot competition. In this case, we would have more entry and price would be lower. The message is that a softer post entry competition induces more entry.

Case III: Strictly convex costs.

Assume that all firms are symmetric and have a strictly increasing smooth convex cost function $C(.)$ and $C(0) = 0$. This implies that average cost is strictly increasing[1]. Assume that all firms setting the lowest price share demand in the same proportion.

There is no asymmetric equilibrium. Observe that in such an equilibrium there would be at least one firm say i that sells the good because it sets the lowest price and another one say j that does not sell the good because it sets a price higher than the lowest one. If firm i obtains negative profits, it can not be an equilibrium configuration because it can increase its profits by setting a price higher than any of its competitors. If firm i obtains nonnegative profits, firm j can obtain positive profits by setting the same price as firm i. Suppose that in the candidate equilibrium m firms set the same price as firm i and nonnegative profits imply

$$\frac{D(p_i)}{m}\left[p - AC(\frac{D(p_i)}{m})\right] \geq 0$$

[1] Assume that $0 < q_1 < q_2$ and define $\lambda = \frac{q_1}{q_2}$. Strict convexity implies that:

$$\begin{aligned} C(\lambda q_2 + (1-\lambda)0) &< \lambda C(q_2) + (1-\lambda)C(0) \\ C(\lambda q_2) &< \lambda C(q_2) \end{aligned}$$

Rearranging we obtain the desired result:

$$\frac{C(q_1)}{q_1} = \frac{C(\lambda q_2)}{\lambda q_2} < \frac{C(q_2)}{q_2}$$

where $AC()$ is average cost. Firm j by deviating and setting p_i obtains:

$$\frac{D(p_i)}{m+1}\left[p - AC(\frac{D(p_i)}{m+1})\right] > 0$$

The strict inequality follows from the fact that average cost is strictly increasing. So in equilibrium all firm set the same price.

To characterize the symmetric equilibrium we define $\pi_n(p) = \frac{pD(p)}{n} - C(\frac{D(p)}{n})$. It is the profit when all the n firms set the price p. The following two conditions should be satisfied in the symmetric equilibrium:

a) $\pi_n(p) \geq \pi_1(p)$ Lowering the price is not profitable. By lowering the price a firm serves the whole demand. It may not be profitable because average cost is increasing and the firm has to satisfy all the forthcoming demand at price p.

b) $\pi_n(p) \geq 0$ Increasing the price is not profitable.

As usual a graphic will help us to determine the equilibrium. We plot π_n and π_1. It is possible to check that they have the following shape. If we denote by $\underline{p}_n$ and $\bar{p}_n$ the prices that satisfy $\pi_n(\underline{p}_n) = 0$ and $\pi_n(\bar{p}_n) = \pi_1(\bar{p}_n)$ we have that all firms setting a price p is an equilibrium if $p \in [\underline{p}_n, \bar{p}_n]$ because then both conditions above are satisfied.

Observe that the competitive price (w_n) is included in the set. As in the case with constant marginal costs we have that the competitive solution can be sustained as an equilibrium of a game where firms choose prices. You will deal with the asymmetric case in the problem set.

So far we have assumed that firms has to supply all the forthcoming demand at a given price. What happens if we suppress this assumption ? The answer is Bertrand-Edgeworth competition.

2.2 Bertrand-Edgeworth competition.

Now we are going to see what happens when firms are not obliged to supply completely the residual demand they face. In other words, firms choose both the price and the quantity they want to sell. This type of market competition is known as Bertrand-Edgeworth competition. At a given price p_i, a firm would like to sell no more than its competitive supply at this price $q_i = S_i(p_i)$.

In this situation, for example in duopoly, it may be the case that the higher priced firm sells a positive amount because the lower-priced firm does not supply the whole demand at its price. In this case, one must specify a rule for rationing demand to determine the amount sold by the higher priced firm. Two leading rules have been proposed: surplus-maximizing and proportional.

We are going to illustrate them in a market with demand $D(p)$ and assuming $p_2 < p_1$. To clarify things it is good to think in the case with many consumers with unit demands.

In any case, firm 2 is going to sell $\min\{D(p_2), S_2(p_2)\}$. If $D(p_2) \leq S_2(p_2)$ all demand is served by firm 2 and there is no residual demand for firm 1. The interesting case arises when $D(p_2) > S_2(p_2)$. Firm 2 serves $S_2(p_2)$ consumers. The question is how are those consumers selected from the ones that want to buy at price p_2. The surplus-maximizing rationing rule selects the ones that value the good the most. The proportional rule selects a random sample, where every consumer is chosen with the same probability $\frac{S_2(p_2)}{D(p_2)}$. If we have a large number of consumers a proportion $\frac{S_2(p_2)}{D(p_2)}$ of all consumers will be selected. This selection determines the residual demand of firm 1.

$D(p_1)$ consumers want to buy at price p_1. They are a subset of the consumers that want to buy at p_2. The important point is that it includes the consumers that value the good the most selected in the surplus maximizing rationing rule. Therefore, in this case, the demand left for firm 1 is $D(p_1) - S_2(p_2)$.

In the proportional case, given that the selection was random, we have that the proportion of consumers selected is the same in the set than in the subset. Therefore, from the consumers that want to buy at price p_1, $\frac{S_2(p_2)}{D(p_2)}$ $D(p_1)$ buy from firm 2 and the residual demand of firm 1 is $(1 - \frac{S_2(p_2)}{D(p_2)})D(p_1)$ whenever it is positive and zero otherwise.

One can check that the residual demand is larger with the proportional rule than with the surplus-maximizing rationing rule.

$$D(p_1) - S_2(p_2) - D(p_1)\left(1 - \frac{S_2(p_2)}{D(p_2)}\right) = -S_2(p_2)\left(1 - \frac{D(p_1)}{D(p_2)}\right) < 0$$

We can see it for the case of a linear demand. Assume that firm 2 sets the price p_2 and sells $S_2(p_2)$. For the case of the surplus-maximizing rationing rule, the demand for prices higher than p_2 is obtained by a parallel displacement such that demand at any such prices is reduced in $S_2(p_2)$. For the proportional rule the demand rotates around the intercept such that for any such prices the proportion of demand served by firm 2 is the same than in p_2. The specific shape of demand for $p > p_2$ is given by:

$$D(p) = (a - P)\left(1 - \frac{S_2(p_2)}{D(p_2)}\right)$$

It has the same intercept that the original demand.

We are going to study a duopoly market when firms face capacity limits (k_1 and k_2, respectively) but otherwise have zero production costs. This implies that $S_i(p) = k_i$.

The equilibrium price with pure strategies should satisfy:

$$p_1 = p_2 = P(k_1 + k_2) \tag{6}$$

This is the competitive price again. The first thing to check is that equilibrium can not be asymmetric. $p_i > p_j > 0$ can not be an equilibrium. If firm j serves the whole demand, (and therefore firm i makes no sales), firm i increases its profits by undercutting firm j. Otherwise, firm j can increase its price without losing sales and therefore increasing profits. $p_i > p_j \geq 0$ can not be an equilibrium, because firm j increases its profits by raising slightly its price.

If $p_1 = p_2 > P(k_1+k_2)$, there is at least one firm that sells less than capacity and that increases profits by lowering slightly the price, increasing in this way its sales. This is the typical undercutting we have in the Bertrand model.

If $p_1 = p_2 < P(k_1+k_2)$, consumers are rationed and then a firm can increase slightly its price without losing sales.

(6) will be an equilibrium if firms do not want to deviate. Certainly, firms do not want to set a lower price because they can not increase sales, as they are already selling up to capacity. (6) will be an equilibrium if they do not want to increase them.

$p_1 = p_2 = P(k_1 + k_2) = 0$ is an equilibrium if $P(k_i) = 0$. Firm j does not want to increase its price, because he will get no demand as firm i serves the whole demand at zero price. Then this type of equilibrium exists if $k_i \geq D(0)$ for $i = 1, 2$.

The conditions such that $p_1 = p_2 = P(k_1 + k_2) > 0$ is an equilibrium change depending on the rationing rule we consider.

With *the surplus-maximizing rationing rule*, if firm i raises its price, its demand will be $D(p_i) - k_j$. But this is exactly the residual demand in the Cournot case. In this case, firm i would like to sell $r(k_j)$. If $k_i > r(k_j)$, it wants to increase the price. Then the equilibrium in pure strategies only exists if $k_i \leq r(k_j)$.

With *the proportional rule,* if firm i raises its price, its residual demand will be $(1 - \frac{k_j}{k_1 + k_2})D(p_i)$. Given that it is the original demand except a constant, the price that maximizes its profits is the monopoly price with zero costs (p^m). Firm i does not want to increase the price if $p^m \leq P(k_1+k_2)$ or $k_1+k_2 \leq D(p^m)$.

PICTURE 2

We can obtain the region with pure-strategy equilibria for the case of linear demands in the space of capacities. We draw the Cournot best-responses of firms. One intercept is obtained by the quantity that induces a firm not to produce. This is the case when the price is already zero and this quantity amounts to $D(0)$. Then, in this region, an equilibrium in pure strategies with zero price exists. The other intercept coincides with the monopoly output. Then the region under the line where the sum of capacities amounts to less than the monopoly output is the one where an equilibrium in pure strategies exist for the proportional rule. Finally, the region under both best-response functions is the one where an equilibrium in pure strategies exist for the surplus-maximizing rule.

So we have a region without equilibria in pure strategies and the only hope is that equilibria in mixed strategies exist. Observe that the region is larger with the proportional rationing rule. The reason is that firms have more incentive to increase the price in this case, given that the residual demand is also larger.

For the general case, it is quite complicated to compute the equilibria in mixed strategies. However its computation is straightforward for the case of symmetric capacities ($k_1 = k_2 = k$) and surplus-maximizing rationing rule. As we are in the mixed strategy region we have that $r(k) < k < D(0)$.

To compute the mixed-strategy equilibrium take as given that there is a symmetric mixed-strategy equilibrium where firms randomize in the support $[\underline{p}, \overline{p}]$ according to an atomless distribution function $F()$.

First of all we are going to find the bounds of the support. In order to do that one should recall that if a price p yields a higher expected profit (given the strategy of the opponent) than another price q, q can not belong to the support. Then we have that the expected profits should be the same in all the prices in the support

$p < P(2k)$ can not belong to the support because it obtains more profits by setting $p+\varepsilon$, independently of the price set by the competitor. If ε is small with both prices it sells up to capacity and therefore it obtains more profits with $p + \varepsilon$. Even if the competitor sets a lower price, demand at p is higher than k. Then if he increases the price to $p + \varepsilon$, he does not lose sales.

$\overline{p} < P(k)$ because if $\overline{p} \geq P(k)$ we have that by setting $\overline{p}$ it will be the firm with the highest price and he will not sell anything. (We should not worry about the case where firms set the same price, because it happens with zero probability). He can obtain positive expected profits by setting $P(k)-\varepsilon$, because then it sells something for sure.

Then for the prices in the support the following condition must hold.

$$P(2k) \leq p < P(k)$$

Then we have that in the prices in the support the firm that sets the lowest price sells up to capacity and the other firm sells less than the capacity. Therefore the expected profits of setting price p can be written as:

$$(1 - F(p))pk + F(p)p(D(p) - k)$$

where $(1 - F(p))$ is the probability that the other firm sets a price higher than p and $F(p)$ is the probability that the other firms sets a price lower than p.

Let us[2] define $p^* = \arg\max_p p(D(p) - k)$. If he sets $\overline{p}$ a player knows that it will be the firm with the highest price and therefore the expected profits will be $\overline{p}(D(\overline{p}) - k)$. We can see that it is not possible that $\overline{p} > p^*$: $\overline{p}$ can not belong to the support because it gives less expected profits than p^* and this is a contradiction. We have that $(1 - F(p^*))p^*k + F(p^*)p^*(D(p^*) - k) > p^*(D(p^*) - k) > \overline{p}(D(\overline{p}) - k)$. The first inequality comes from the fact that the firm that sets the high price sells less than the capacity and the second from the definition of p^*. We can see that it is not possible that $\overline{p} < p^*$. We have that $p^*(D(p^*) - k) > \overline{p}(D(\overline{p}) - k)$. Therefore, $\overline{p} = p^*$.

[2] Observe that $0 < D(p^*) - k = r(k) < k$, because we are in the mixed strategy region. Then $P(2k) < p^* < P(k)$.

Define $\bar{\pi} = p^*(D(p^*) - k)$. This should be the expected profits of all prices in the support. If a firm sets a price $\underline{p}$, with probability 1 he will set a price lower than the competitor. Then:

$$\underline{p} = \frac{\bar{\pi}}{k}$$

We have already the bounds of the support. For any price in the support, expected profits should be equal to $\bar{\pi}$:

$$\bar{\pi} = (1 - F(p))pk + F(p)p(D(p) - k)$$

$F(p)$ in this equation gives us the distribution function of prices in equilibrium:

$$F(p) = \frac{k - \frac{\bar{\pi}}{p}}{2k - D(p)}$$

Observe that the expected profits amount to $p^*(D(p^*) - k)$. In terms of inverse demands and best responses we have that it amounts to:

$$P(k + r(k))r(k)$$

It amounts to the profits that a firm would obtain when the competitor sells all its capacity. When capacities are asymmetric[3] $k_i > k_j$, we obtain the same result in the sense that firm i exactly obtains in expected terms:

$$P(k_j + r(k_j))r(k_j)$$

This will be important for what we are going to see next.

2.3 Sequential choice of capacity and prices.

As we have obtained the equilibrium payoff of a game with given capacities, we can analyze now a previous stage where firms choose capacities. The cost of creating capacity is given by ck_i. We know that the only complicated part of the previous game was that for large capacities there was no equilibrium in pure strategies. To avoid this problem we can assume a large enough c so that it can never be profitable to build such large capacities. For example, with linear demand $p = a - Q$, a necessary condition for the capacity choice to be profitable is that $(a^2/4) - ck_i \geq 0$, otherwise it would be impossible to recover the investment. This means that firms only want to choose capacities satisfying $k_i \leq a^2/4c$. If $c \geq 3a/4$, then $k_i \leq a/3$. For this level of capacities pure-strategy equilibria exists with payoffs:

$$(a - (k_1 + k_2) - c)k_i$$

[3] As we are in the region of mixed-strategy equilibria, we have that $k_i > r(k_j)$.

The same payoffs as in the Cournot game and therefore we will have the same equilibrium

$$k_1 = k_2 = k^c = \frac{a-c}{3}$$

Summarizing "Quantity pre-commitment and Bertrand competition yield Cournot outcomes". This is the title of the paper of Kreps and Scheinkman (1983). They show that the same equilibrium is obtained even when c is low. The reason is that when every firm chooses the Cournot capacity no firm wants to deviate. (k^c, k^c) is located in the pure-strategy equilibrium region because when the unit cost increases, best-responses move downwards. No firm wants to deviate choosing a capacity in this region because we have the same payoffs as in the Cournot case. May it be profitable to choose a large enough capacity $k_i > r(k^c)$ so that we are in the mixed strategy equilibrium region ? The answer is NO. In this case, by doing so, a firm would be the firm with the largest capacity and as we have seen before he will obtain as expected profits:

$$P(k^c + r(k^c))r(k^c) - ck_i$$

where $r()$ is the best response with zero costs. Then we have

$$P(k^c + r(k^c))r(k^c) - ck_i < P(k^c + r(k^c))r(k^c) - cr(k^c) < P(k^c + k^c)k^c - ck^c$$

The first comes from the fact that $k_i > r(k^c)$. The last inequality comes from the fact that

$$k^c = \arg\max_q P(k^c + q)q - cq$$

2.4 Price competition with differentiated goods.

We have studied so far two different types of competition (price and quantity competition) in markets of homogenous goods. We have checked that both types of competition yielded very different market outcomes. For example, in the case of symmetric firms and constant marginal costs, with Cournot competition the competitive outcome is obtained in the limit as the number of firms tends to infinity whereas it is already obtained with only two firms with Bertrand competition. We are going to check whether those differences persist when goods firms sell are differentiated. We are going to assume the existence of a demand system with differentiated goods. Rationalizations from where differentiation comes from will be studied in the chapter specifically devoted to Product Differentiation.

We are going to focus on the following example of inverse demands with two goods:

$$\begin{aligned} \alpha - \beta q_i - \gamma q_j &= P_i \\ \text{where } \beta &> \gamma \geq 0 \end{aligned}$$

Goods are substitutes. $\frac{\gamma}{\beta}$ expresses the degree of product differentiation, ranging from 0 for independent goods to 1 for perfect substitutes. Inverting the system we obtain direct demands. They are given by:

$$\begin{aligned} q_i &= a - bp_i + cp_j \\ \text{where } a &= \frac{\alpha}{\beta+\alpha},\ b = \frac{\beta}{\beta^2-\gamma^2} \text{ and } c = \frac{\gamma}{\beta^2-\gamma^2}. \end{aligned}$$

We can solve the Cournot equilibrium and the Bertrand equilibrium assuming that each good is produced by a different firm. Each firm produces at constant marginal cost c. In the Cournot case, we have that

$$\pi_i = (\alpha - c - \beta q_i - \gamma q_j)q_i$$

The Cournot equilibrium is the solution of the following system ($i, j = 1, 2$ and $i \neq j$) :

$$\frac{\partial \pi_i}{\partial q_i} = \alpha - c - 2\beta q_i - \gamma q_j = 0 \tag{7}$$

The solution is:

$$\begin{aligned} q_1 &= q_2 = q^C = \frac{\alpha - c}{2\beta+\gamma} \\ P^C &= \frac{\alpha\beta + (\beta+\gamma)c}{2\beta+\gamma} \end{aligned}$$

Observe that (7) yields a best-response function that is downward sloping:

$$r_i(q_j) = \frac{\alpha - c - \gamma q_j}{2\beta}$$

It is said in this case that we have strategic substitutes: an increase in the output of a firm, decreases the optimal output of the other firm. We can obtain the equilibrium graphically representing the best-responses.

In the Bertrand case, we have:

$$\pi_i = (a - bP_i + cP_j)(P_i - c)$$

The Bertrand equilibrium is the solution of the following system ($i, j = 1, 2$ and $i \neq j$) :

$$\frac{\partial \pi_i}{\partial P_i} = a - 2bP_i + cP_j + bc = 0 \tag{8}$$

The solution is:

$$P_1 = P_2 = P^B = \frac{a+bc}{2b-c} = \frac{\alpha(\beta-\gamma)+\beta c}{2\beta-\gamma}$$

Observe that (8) yields a best-response that is upward sloping:

$$r_i(P_j) = \frac{a + cP_j + bc}{2b}$$

It is said in this case that we have strategic complements: an increase in the price of a firm, increases the optimal price of the other firm. This is a very important difference between the Cournot and the Bertrand model that explains that very often they yield very different predictions. We will find examples of this divergence in the next chapter. We can obtain the equilibrium graphically representing the best-responses.

Another difference is that Bertrand yields a more competitive outcome than Cournot. Indeed we have:

$$P^C - P^B = \frac{\gamma^2(\alpha - c)}{4\beta^2 - \gamma^2} > 0$$

The reason of this result (that can be proved for a general set of demands) is that residual demands (the demand -combination of prices and quantities- that is left to a firm given the strategy of competitors) are much flatter in Bertrand than in Cournot. In other words, price cuts are much more effective in Bertrand to gain demand. In the Bertrand case you are not only able to attract new customers that want to benefit from lower prices BUT YOU ARE ALSO ABLE TO STEAL CUSTOMERS FROM YOUR COMPETITORS. This is not possible in Cournot because the sales of competidors are fixed. In other words, in Cournot "each firm expects the other to cut prices in response to price cuts (to keep their sales constant) while in Bertrand competition the firms expect the other to maintain prices".

In the Bertrand case residual demand is simply $q_i = a - bp_i + c\bar{p_j}$ and in the Cournot case $p_i = \alpha - \beta q_i - \gamma\ \bar{q_j}$ or $q_i = \dfrac{\alpha - p_i - \gamma\bar{q_j}}{\beta}$. The demand is flatter in the Bertrand case because we have that $b > \frac{1}{\beta}$. We already checked this for the case of homogenous goods.

3 A General Model of two-stage games.

So far, we have considered situations where firms took decisions simultaneously. In this chapter we are going to consider decisions that are taken sequentially. We can think in the situation that before taking the decisions in the market, another set of decisions are taken that indirectly affect the outcome in the market stage. In particular, we are going to study games where n firms take decisions in two stages. Call t_i the choice taken by firm i in the first stage and s_i its choice in the second stage. If s and t are a vector of decisions of all firms in the second and first stage respectively, we have that the profit of firm i can be written as $\pi_i(s,t)$. We are making the very important assumption that decisions taken in the first stage can not be revised in the second stage and that they are observed by all firms. In other words, a firm when takes its decisions in the second stage knows the decisions taken by all firms in the first stage. In this case, a strategy for firm i consists of a decision in the first stage t_i and a decision in the second stage as a function of all possible decisions taken in the first stage $s_i(t)$ for any t.

We will use as a solution concept the subgame perfect Nash equilibrium. A set of strategies is a subgame perfect Nash equilibrium if they conform a Nash equilibrium for any subgame. In our case, $t^* = (t_1^*, ..t_n^*)$ and $s^*(t) = (s_1^*(t), ..s_n^*(t))$ is a subgame perfect Nash equilibrium if:

(a) $s^*(t)$ is a Nash equilibrium of the game defined by the choice of t in the first stage.

(b) For all i, $t_i^* = \arg\max \pi_i(s^*(t_i, t_{-i}^*), t_i, t_{-i}^*)$.

If both decisions can be expressed as the choice of a real number and functions are well behaved, the two previous conditions can be written as:

(a) For any t and i

$$\frac{\partial \pi_i}{\partial s_i}(s^*(t), t) = 0 \tag{9}$$

(b)

$$\frac{\partial \pi_i}{\partial t_i}(s^*(t), t^*) + \sum_{j=1}^{n} \frac{\partial \pi_i}{\partial s_j}(s^*(t), t^*) \frac{\partial s_j^*}{\partial t_i} = 0$$

Given (9), can be rewritten as:

$$\frac{\partial \pi_i}{\partial t_i}(s^*(t), t^*) + \sum_{j \neq i} \frac{\partial \pi_i}{\partial s_j}(s^*(t), t^*) \frac{\partial s_j^*(t)}{\partial t_i} = 0$$

The first term represents the *direct effect* of t_i on profits, whereas the second term represents its effect on profits through changing the strategies that competitors are going to play in the second stage. This second effect is called the *strategic effect*.

There are no general properties of two-stage games. Therefore, the rest of this chapter will be devoted to present different examples. The main purpose of the following simplifications is to be able to relate the sign of the strategic effect with the slope of the reaction function of firms.

3.1 Cost reducing investement.

Assume we have two firms. Both firms compete in the market selling goods that are substitutes. Previous to the market competition stage, firm 1 can invest in reducing its marginal cost of production. This investment is denoted by K_1.

The second stage equilibrium given a level of investment K_1 is given by:

$$s_1^*(K_1) \text{ and } s_2^*(K_1)$$

Then the objective of Firm 1 in the first stage is given by:

$$\pi_1(K_1, s_1^*(K_1), s_2^*(K_1))$$

The optimal value of the investment will be given by the following first order condition:

$$\frac{d\pi_1}{dK_1} = \frac{\partial \pi_1}{\partial K_1} + \frac{\partial \pi_1}{\partial s_2} \frac{ds_2^*}{dK_1}$$

The equilibrium conditions of the second stage allow us to ignore the effect on π_1 on the change in firm 1's second-period action. The important step is the one that allows us to write the strategic effect as a function of the slope of the second-period reaction curve. We have that:

$$s_2^*(K_1) = R_2(s_1^*(K_1))$$

where $R_2()$ is the second stage reaction function of firm 2. Then deriving we have:

$$\frac{ds_2^*}{dK_1} = [R_2'(s_1^*)]\frac{ds_1^*}{dK_1} \tag{10}$$

We can sign the strategic effect (SE) for the two polar cases of Bertrand and Cournot competition.

$$SE = \frac{\partial \pi_1}{\partial s_2}[R_2'(s_1^*)]\frac{ds_1^*}{dK_1}$$

With Cournot competition, we have that increases in the output of the competitor reduces profits, reaction function is downward sloping and the more you invest the lower the cost and higher the production. Therefore the strategic effect is positive. This means that the firm invests more than the profit-maximizing level.

$$\begin{array}{rcl} SE & = & \dfrac{\partial \pi_1}{\partial s_2}[R_2'(s_1^*)]\dfrac{ds_1^*}{dK_1} > 0 \\ & & (-) \quad\; (-) \quad\; (+) \end{array}$$

With Bertrand competition, we have that increases in the price of the competitor increases profits, reaction functions are upward sloping and the more you invest the lower the cost and lower the price. Therefore the strategic effect is negative. This means that the firm invests less than the profit-maximizing level.

$$\begin{array}{rcl} SE & = & \dfrac{\partial \pi_1}{\partial s_2}[R_2'(x_1^*)]\dfrac{ds_1^*}{dK_1} < 0 \\ & & (+) \quad\; (+) \quad\; (-) \end{array}$$

3.2 The Stackelberg model.

We proceed to revisit the duopoly situation but assume that a firm takes its decisions (the leader) before the other one (the follower). The model can be generalized to more than two firms. We concentrate on the duopoly case to allow for a graphical representation. We assume that the goods are substitutes.

We start with the case of quantity competition. Firm 1 (2) is the leader (follower). In this case, $t_1 = q_1, t_2 = \emptyset$ and $s_1 = \emptyset$ and $s_2 = q_2$.

$$\frac{\partial \pi_2}{\partial q_2}(s_2^*(q_1), q_1) = 0$$

implies that $s_2^*(q_1) = R_2(q_1)$. The objective of the leader in the first stage is given by:

$$\pi_1(R_2(q_1), q_1)$$

And then its optimal decision will satisfy:

$$\frac{\partial \pi_1}{\partial q_1}(R_2(q_1^*), q_1^*) + \frac{\partial \pi_1}{\partial q_2}(R_2(q_1^*), q_1^*) R_2'(q_1^*) = 0$$

The previous equation evaluated at the Cournot output is positive because the first term is zero and the second (the strategic effect) is positive: reaction functions are downward sloping and an increase in the output of the competitor reduces profits because it depresses price. Therefore, the leader will choose an output greater than the one chosen in the equilibrium of the simultaneous move game.

We can obtain the same result graphically using the isoprofit curves. They represent the outputs that give the same profit level to a firm.

$$\pi_i = \bar{\pi}$$

$$\frac{dq_j}{dq_i} = -\frac{\partial \pi_i / \partial q_i}{\partial \pi_i / \partial q_j}$$

As we have seen that the denominator is negative, the sign of the ratio will be the sign of the numerator. It is simply marginal profitability. It is positive for $q_i < r(q_j)$, zero for $q_i = r(q_j)$ and negative for $q_i > r(q_j)$. To draw it we need to draw first the best response. Utility increases as the competitor decreases its output and we approach the monopoly output. How can we identify the output chosen by the leader? The isoprofits we have just drawn correspond to the leader. We need to draw the best response of the follower. The leader will choose the point on the best response of the follower that gives him more profits. In this point the isoprofit is tangent to the best response of the follower. Then we obtain that the output chosen by leader is greater than the output he chooses in the simultaneous-move equilibrium. As a result of this, the follower obtains less profits than in the simultaneous-move equilibrium. Of course, the leader obtains more profits, because he could replicate the situation in the simultaneous-move game.

For the case of Bertrand competition, we have that $t_1 = p_1, t_2 = \emptyset$ and $s_1 = \emptyset$ and $s_2 = p_2$.

$$\frac{\partial \pi_2}{\partial p_2}(s_2^*(p_1), p_1) = 0$$

implies that $s_2^*(p_1) = R_2(p_1)$. And then,

$$\frac{\partial \pi_1}{\partial p_1}(R_2(p_1^*), p_1^*) + \frac{\partial \pi_1}{\partial p_2}(R_2(p_1^*), p_1^*) R_2'(p_1^*) = 0$$

The previous equation evaluated at the Bertrand prices is positive because the first term is zero and the second (the strategic effect) is positive: best response is upward sloping and an increase in the price of the competitor increases profits because it increases its demand. Therefore, the leader will choose a greater price than the one chosen in the the equilibrium of the simultaneous move game.

Again we can obtain the same result graphically by using isoporofit curves. They represent the prices that give the same profit level to a firm.

$$\pi_i = \bar{\pi}$$

$$\frac{dp_j}{dp_i} = -\frac{\partial \pi_i / \partial p_i}{\partial \pi_i / \partial p_j}$$

We have seen that the denominator is positive. The sign of the ratio will be different than the sign of the numerator. It is simply marginal profitability. It is positive for $p_i < r(p_j)$, zero for $p_i = r(p_j)$ and negative for $p_i > r(p_j)$. To draw it we need to draw first the best response. Utility increases as the price of the competitor increases and we move away from the origin. To identify the price chosen by the leader we draw the best response of the follower. The leader will choose the point on the best response of the follower that gives him more profits. In this point the isoprofit is tangent to the best response of the follower. Then the price chosen by the leader is greater than the price he chooses in the simultaneous move equilibrium. As a result of this, the profits of the follower increases with respect to the level he attains in the simultaneous move equilibrium because a higher price of the competitor means more demand. Of course, the leader obtains more profits, because he could replicate the situation in the simultaneous-move game.

3.3 Delegation games.

A spetial type of the two-stage games we are analyzing arises when the decisions taken in the first stage have no direct effect on profits. Its only effect comes through modifying the equilibrium to be played in the second stage. A typical case where this situation arises is when decisions in the first and the second stage are taken by different agents. The decision in the first stage has the only effect of determining the payoff of the agent taken second stage decisions and therefore it affects the equilibrium that is to be obtained. Then t_i is chosen to maximize $\pi_i(s(t))$ while s_i maximizes $\pi_i^t(s)$. In our case, $t^* = (t_1^*, ..t_n^*)$ and $s^*(t) = (s_1^*(t), ..s_n^*(t))$ is a subgame perfect Nash equilibrium if :

$$\frac{\partial \pi_i^t}{\partial s_i}(s^*(t)) = 0$$

$$\sum_{j=1}^{n} \frac{\partial \pi_i}{\partial s_j}(s^*(t)) \frac{\partial s_j^*}{\partial t_i} = 0 \tag{11}$$

To be able to use the reaction functions we restrict attention to the case of 2 firms and only firm 1 makes its decisions in Stage 1. The second stage equilibrium is given by:

$$s_1^*(t_1) \text{ and } s_2^*(t_1)$$

Then (11) can be written for firm 1 as:

$$\frac{\partial \pi_1}{\partial s_1}\frac{ds_1^*}{dt_1} + \frac{\partial \pi_1}{\partial s_2}\frac{ds_2^*}{dt_1} = 0$$

Using the fact that

$$\begin{aligned} s_2^*(t_1) &= R_2(s_1^*(t_1)) \\ \frac{\partial s_2^*}{\partial t_1} &= R_2'(s_1^*)\frac{ds_1^*}{dt_1} \end{aligned}$$

, we have:

$$\frac{ds_1^*}{dt_1}[\frac{\partial \pi_1}{\partial s_1} + \frac{\partial \pi_1}{\partial s_2}R_2'(s_1^*)] = 0$$

This expression gives information about the t_1^* that will be chosen whenever the second term of the expression in brackets has a well defined sign. If it is positive (negative) $\frac{\partial \pi_1}{\partial s_1} < (>)0$, means that t_1 will be selected to induce the agent taking decisions in the second stage to choose an action greater (smaller) than the one that would maximize profits

In Cournot the second term is positive. Therefore, the agent that maximizes profits wants that an ouput greater than the one that maximizes profits is chosen in the second stage, in order to reduce the output sold by competitors.

In Betrand the second term is also positive. Therefore, the agent that maximizes profits wants that a price greater than the one that maximizes profits is chosen in the second stage, in order to increase the price set by competitors.

We are going to present different models of the literature where the equilibrium can be characterized using the observation we have just made. Those papers have the merit of rationalizing common practices that look wrong when the strategic perspective is not taken into account.

- *Export subsidies.* Assume that a firm of a given country compete in the international market with other foreign firms. Before competition takes place, the government of this country can give to the firm an export subsidy. The governemnt wants to maximize the profits of the national firm. We know that if competition is à la Cournot, the government would like to stimulate the production of the firm and this is achieved by giving him a positive subsidy. If instead competition is à la Bertrand, the government wants to induce the firm to set a price greater than the one that would maximize profits and this can be achieved by giving the firm a negative subsidy (Brander and Spencer (1984)). The papers rationalize the fact that Governments do give export subsidies to their national firms. Unfortunately, the "nice" result is only obtained for the case of Cournot competition.

- *Vertical separation.* Assume that a firm competing with other firms makes its sales through an independent retailer. How will the two-part tariff supply contract offered to the retailer be ? The contract includes a payment per unit and a fixed fee. The firm wants to maximize the joint profits of distribution and production because the firm is able to extract all the rents from the retailer through the fixed fee. The firm has to induce the retailer to sell more than the quantity that maximizes profits if competition is à la Cournot and to set a price higher than the profit-maximizing one if competition is à la Bertrand. The first objective is attained by setting a per unit payment lower than marginal cost and the second objective by raising the per unit payment over marginal cost. In the first case, it is like if the firm were offering discounts to the retailer (Bonano and Vickers (1988))

- *Strategic managerial incentives.* The owner of a firm delegates decisions to a professional manager. The owner gives its manager an incentive contract which is a linear combination of profits (Π) and sales (S): $\alpha\Pi+(1-\alpha)S$ plus a fixed fee (F). The owner extracts all the rents through the fixed fee and therefore it will choose α to maximize profits. Observe that α determines the part of the cost that the manager internalizes, because the incentive contract can be rewritten as $O = S - \alpha C$, where C are the costs. In Cournot, output should be stimulated by reducing the costs perceived by the manager and setting $\alpha < 1$. In Bertrand, to induce the manager to set a price higher than the one that maximizes profits the firm should increase the costs perceived by the manager by setting $\alpha > 1$. (Vickers (1985), Fershtman and Judd (1987) and Sklivas (1987)). These results (again for the Cournot case) have two nice implications. On the one hand, the fact that the firms maximize profits is usually defended as a suitable assumption on natural selection grounds in the sense that firms that do not maximize profits should be driven out of the market. These papers attack this idea because they show that firms that do not maximize profits can obtain more profits than profit-maximizing firms. On the other hand, the business literature warns that the fact that managers objective depart from profit maximization to include size considerations may have a negative impact on stockholders wealth. The present papers instead defend that this departure may be good for stockowners because it may increase the profits of the firms (observe that $\alpha < 1$ implies that managers care not only about profits but also on sales that can be used as a proxy of size).

- *Divisionalization.* A firm may decentralize the output and sales decisions by creating divisions (quasi-firms) that maximize their own profits. Assume that the firm can extract all the rents from the divisions so that he maximizes the joint profits of the divisions he has created. The higher the number of divisions the higher their output. Stimulating output has a positive effect if competition is à la Cournot but a negative effect if competition is à la Bertrand. Therefore in the fomer case it will create independent divisions whereas in the latter case the firm will not create any division. The internal organization of a firm (in this case whether it has divisions or decisions are taken centrally) has been usually justified by using efficiency arguments. In the present model the explanation is purely strategic i.e. what counts is the effect divisionalization has on the behavior of competitors. It has a positive effect with Cournot competition because competitors react by reducing their output, but a negative effect with Bertrand competition because competitor react by reducing their price. (Baye, M., K. Crocker, and J. Ju, 1996, Divisionalization, Franchising and Divestiture Incentives in Oligopoly, American Economic Review, 86, 223-236.Corchón, L., 1991, Oligopolistic Competition among Groups, Economics Letters, 36, 1-3. Corchón, L. and M. González-Maestre, 2000, On the competitive effects of divisionalization, Mathematical Social Sciences, 39, 71-79.)

- *Loan commitments*. Assume now that production should be financed and that the interest rate is r. Then the total cost of a firm amounts to $(1+r)C_i(q_i)$. The interest rate affects the marginal cost of a firm. Assume that in an earlier stage, the firm may sign a contract with a bank stipulating that he will be financed at an interest rate r_i. The lower r_i the greater the fixed fee that the firm should pay to sign the contract so that total cost remains unchanged. The only effect of signing the contract is that it changes the marginal cost in the market stage. If $r_i < r$ the contract stimulates the production of the firm because it reduces the marginal cost whereas the opposite effect is obtained if $r_i > r$. The former effect is the optimal with Cournot competition and therefore the firm will sign a financial contract with $r_i < r$. The latter effect is the optimal with Bertrand competition and therefore the firm will sign a financial contract with $r_i > r$. (Maksimovic (1990)).

One must say that we get multiplicity of equilibria when the strategy space in the first stage becomes more complex. For example, Kühn (1997) assumes that we have two firms that sell the good through exclusive retailers. In the first stage firms decide the supply contract and in the second stage retailers decide the level of sales. The supply contract is a function $W(q_i)$ determining the payment given a quantity supplied. The only restriction on $W(q_i)$ is that it has to be a continuous function. Then any decreasing function can arise as the reaction function of the retailer in the second stage if the supply contract is chosen appropriately. Observe that supply contracts only generate decreasing reaction functions. Profits of the retailer are:

$$\pi_i = P(q_1 + q_2)q_i - W(q_i)$$

The best response is defined implicitly in the FOC:

$$\frac{\partial \pi_i}{\partial q_i} = P(q_1 + q_2) + P'(q_1 + q_2)q_i - W'(q_i) = 0$$

The slope of the best response is given by:

$$R'(q_j) = -\frac{\dfrac{\partial^2 \pi_i}{\partial q_i \partial q_j}}{\dfrac{\partial^2 \pi_i}{(\partial q_i)^2}}$$

The denominator is negative because the firm is maximizing. The numerator is not affected by $W(q_i)$ and it is negative given the usual assumptions on demand.

We are going to check that this generates multiplicity of equilibria. We draw the reaction functions if firms sold directly to consumers. (We assume that through the contract firms extract all the rents from the retailers, so that they maximize the joint profits obtained in manufacturing and distribution). Then we can draw the isoprofits of firms. The equilibrium condition is that isoprofits should be tangent to the reaction function of the reatiler of the rival firm. Take a point over the reaction functions. Draw the isoprofits (one for each firm) passing through it. Next we can draw two (decreasing) lines tangent to each of the isoprofits curves. As they are decreasing, there exists contracts such that those lines correspond to reaction functions of retailers. Those contracts are equilibrium contracts.

This can not happen in the region under the reaction functions, because at least for one firm isoporofits are upward sloping and then they can not be tangent to the decreasing reaction function of the rival retailer. Although we have multiplicity, we have the clear prediction that price should be lower than the price if firms sold directly to consumers. It is the same conclusion as the one in the model with simply two-part tariff contracts.

Assume that demand is given by $P(Q+\theta)$ where θ is a negative shock of demand. [In the linear case we would have that demand is $P=\alpha-\theta-Q$, demand is lower as θ increases]. Uncertainty reduces the number of equilibria because it increases the number of conditions to be satisfied by equilibrium contracts. Under certainty, the equilibrium contract should generate a reaction function that cuts the one of the rival retailer in the point where profits are maximized. Under uncertainty this condition should be satisfied for any realization of demand. Kühn (1997) shows that when the support of the shock is not bounded below and demand is linear we have an unique equilibrium. There is only one contract that appropriately adjusts to any realization of the demand shock

3.4 Imperfect Observability

All we have said so far assumed observability of the decisions taken in the first stage. If observability is assumed away is like if decisions were taken simultaneously and the strategic effects disappear. For example, we can analyze the Stackelberg model if the actions taken by the leader are not observed by the follower. In this case the strategy of the follower would be a level of production and not an output as a function of the production of the leader. In equilibrium strategies should satisfy:

$$q_1^* =^{\arg\max}_{q_1} \pi_i(q_1, q_2^*)$$
$$q_2^* =^{\arg\max}_{q_2} \pi_i(q_2, q_1^*)$$

Those are the equilibrium conditions of the simultaneous-move game.

Now we know what happens in the extreme cases of observability and non-observability. What will happen in the intermediate cases i.e. when we have imperfect observability. Assume (Bagwell (1995)) that the follower observe the production of the leader with a noise. The leader observes $s = q_1 + \varepsilon$, where $\varepsilon \sim N(0, \sigma_\varepsilon^2)$. Given the assumption that the noise is distributed according to a normal, all signals have positive probability given an output. In other words, observing a signal we can not exclude that a given level of production has taken place. What changes is the probability of this fact.

To find the equilibrium, the key is to note that equilibrium requires that $q_2^*(s)$ be a best response to q_1^* for any s, and hence the follower will play $q_2^*(s) = R_2(q_1^*)$, regardless of s. The strategy does not depend on s and, therefore, the leader will not make an effort to try to affect the behavior of the follower by modifying the signal. In equilibrium we must have that $q_1^* = R_1(q_2^*(s))$. Then we have the Cournot quantities. The key point is to understand that knowing the equilibrium strategy of the rival, that in pure strategies means to know the action that he has taken, it is sufficient to derive the optimal strategy and, therefore, you do not pay any attention to the signal s .

This model presents a very strong discontinuity. As the noise of the signal tends to zero ($\sigma_\varepsilon^2 - > 0$) the equilibrium in pure strategies is still the one with simultaneous moves. However when the noise has completely disappeared ($\sigma_\varepsilon^2 = 0$) we switch to the Stackelberg equilibrium. Maggi (1999) builds a model where we do not have this discontinuity. He assumes that the leader has private information on its cost θ. The leader knows its value while the follower only knows that is normally distributed (independently of ε) $\theta \sim N(\bar{\theta}, \sigma_\theta^2)$. To simplify assume that the unit cost of the follower is $\bar{\theta}$ and demand is linear $P = 1 - Q$.

Two benchmark situations will be the equilibrium with sequential choice and perfect observability and the equilibrium with simultaneous choices. We will see that given a level of private information $\sigma_\theta^2. > 0$, they will appear respectively in the model as the limit situations when the noise tends to zero $\sigma_\varepsilon^2 - > 0$ and when the noise tends to infinity. Therefore, the discontinuity will disappear. Observe that the game with simultaneous choice is a game of incomplete information and we use the bayesian equilibrium as the solution concept. We quickly solve these two benchmark models.

Simultaneous game

It is a game of incomplete information and we use the Bayesian equilibrium as the solution concept. To find the equilibrium we have to find a level of output by the follower q_2 and a level of output as a function of the realization of the cost for the leader $q_1(\theta)$ such that both firms maximize their profits. The follower maximizes its expected profits given that the output of the leader is a random variable:

$$E\{(1 - q_2 - q_1(\theta) - \bar{\theta})q_2\} = (1 - q_2 - Eq_1(\theta) - \bar{\theta})q_2$$

Then the optimal output of the leader is:

$$q_2 = \frac{1 - Eq_1(\theta) - \bar{\theta}}{2} \tag{12}$$

To compute $Eq_1(\theta)$ we have to study the behavior of the leader. It maximizes its profits given a realization of the cost θ

$$(1 - q_1 - q_2 - \theta)q_1$$

The optimal output is then:

$$q_1(\theta) = \frac{1-\theta-q_2}{2} \tag{13}$$

Then we have:

$$Eq_1(\theta) = \frac{1-\bar{\theta}-q_2}{2} \tag{14}$$

Substituting (14) in (12) we obtain the equilibrium output of the follower:

$$q_2 = \frac{1-\bar{\theta}}{3} \tag{15}$$

Substituting (15)in (13) we obtain the equilibrium strategy of the leader:

$$q_1(\theta) = \frac{2+\bar{\theta}}{6} - \frac{\theta}{2}$$

Sequential game:

If the leader produces q_1, the follwer will produce $q_2 = \dfrac{1-\bar{\theta}-q_1}{2}$. Then the leader will maximize in the first stage:

$$(1 - \left(\frac{1-\bar{\theta}-q_1}{2}\right) - q_1 - \theta)q_1$$

whose maximizer is:

$$q_1 = \frac{1+\bar{\theta}}{2} - \theta$$

We proceed to solve the equilibrium of the model. The strategy of the leader is a level of production as a function of the realization of its cost: $q_1(\theta)$. The strategy of the follower is a level of production as a function of the signal s: $q_2(s)$. In equilibrium, they should maximize respectively:

$$E\{(1-q_1-q_2(s)-\theta)q_1/\theta\} = (1-q_1-E[q_2(s)/\theta]-\theta)q_1 \tag{16}$$

$$E\{(1-q_2-q_1(\theta)-\bar{\theta})q_2/s\} = (1-q_2-E[q_1(\theta)/s]-\bar{\theta})q_2 \tag{17}$$

We are going to look for an equilibrium in linear strategies:

$$\begin{aligned} q_1(\theta) &= A + B\theta \\ q_2(s) &= \alpha + \beta s \end{aligned}$$

This will allow us to compute the expectations we had in the expected profits:

$$E[q_2(s)/\theta] = E[\alpha+\beta(q_1+\varepsilon)/\theta] = E[\alpha+\beta(q_1+\varepsilon)] = \alpha+\beta q_1+\beta E[\varepsilon] = \alpha+\beta q_1 \tag{18}$$

As firm 1 knows q_1, knowledge of θ does not improve the estimation of competitor's output.

$E[q_1(\theta)/s]$ is the most difficult to compute and one crucially uses the assumption on normality that "has the highly desirable feature of implying linear updating rules" (Judd and Riordan (1994)). We note that:

$$q_1(\theta) = A + B\theta \sim N(A + B\bar{\theta}, B^2\sigma_\theta^2)$$

$$s = A + B\theta + \varepsilon \sim N(A + B\bar{\theta}, B^2\sigma_\theta^2 + \sigma_\varepsilon^2)$$
$$cov(A + B\theta, A + B\theta + \varepsilon) = B^2\sigma_\theta^2$$

If X and Y are jointly normal,

$$E(X/Y) = EX + [cov(X,Y)/\sigma_Y^2](Y - EY)$$

Then,

$$\begin{aligned} E[q_1(\theta)/s] &= A + B\bar{\theta} + \frac{B^2\sigma_\theta^2}{B^2\sigma_\theta^2 + \sigma_\varepsilon^2}(s - A + B\bar{\theta}) \\ E[q_1(\theta)/s] &= (A + B\bar{\theta})(1 - \frac{B^2\sigma_\theta^2}{B^2\sigma_\theta^2 + \sigma_\varepsilon^2}) + \frac{B^2\sigma_\theta^2}{B^2\sigma_\theta^2 + \sigma_\varepsilon^2}s \end{aligned}$$

$$\begin{aligned} E[q_1(\theta)/s] &= \frac{V}{B^2 + V}(A + B\bar{\theta}) + \frac{B^2}{B^2 + V}s \qquad (19) \\ \text{where } V &= \frac{\sigma_\varepsilon^2}{\sigma_\theta^2} \end{aligned}$$

This expression is key to understand the result. As the noise of the signal increases, one pays less attention to it. As the uncertainty on the costs of the leader increases, more weight is given to the signal because the output of the leader becomes more variable. Notice that the more sensitive is the output of the leader to its cost (the greater is B), more weight is given to the signal, because the estimation a priori is worse.

Substituting (18) in (16), we can compute the optimal strategy of the leader:

$$(1 - q_1 - E[q_2(s)/\theta] - \theta)q_1 = (1 - q_1 - \alpha - \beta q_1 - \theta)q_1$$

From the FOC, we can obtain the optimal output of the leader:

$$q_1 = \frac{1 - \alpha - \theta}{2(1 + \beta)} = A + B\theta$$

Then,

$$A = \frac{1 - \alpha}{2(1 + \beta)} \qquad (20)$$

$$B = -\frac{1}{2(1+\beta)} \tag{21}$$

Substituting (19) in (17), we can compute the optimal strategy of the follower.

$$q_2 = \frac{1 - E[q_1(\theta)/s] - \bar{\theta}}{2} = \alpha + \beta s$$

Then,

$$\alpha = (1/2)[1 - \bar{\theta} - \frac{V}{B^2+V}(A + B\bar{\theta})] \tag{22}$$

$$\beta = -(1/2)\frac{B^2}{B^2+V} \tag{23}$$

From (21) we have:

$$\beta = -1 - \frac{1}{2B} \tag{24}$$

Substituting (24) in (23) and rearranging terms we obtain:

$$B^3 + 2VB + B^2 + V = 0 \tag{25}$$

The solution $B(V)$ is increasing and $B(0) = -1$ and $B(\infty) = -1/2$. The remaining coefficients can be determined in terms of B. Given that from (25)

$$V = -\frac{B^3 + B^2}{2B+1}$$

we have that:

$$\alpha = -\frac{B(1-\bar{\theta})}{1-B}$$

$$A = -\frac{B(1-B\bar{\theta})}{1-B}$$

Then finding the equilibrium is over and we can state the main result of the paper.

Proposition 1 *As $V -> 0$, the leader's output approaches the Stackelberg output. As $V -> \infty$, the leader's output approaches the simultaneous-move output.*

If $B = -1$, $q_1 = \dfrac{1+\bar{\theta}}{2} - \theta$ that is the output of the leader in the perfect information Stackelberg outcome. If $B = -\frac{1}{2}$, $q_1 = \dfrac{2+\bar{\theta}}{6} - \dfrac{\theta}{2}$ that is the output of the leader in the simultaneous-move (Nash-Bayes) outcome.

4 Horizontal and Vertical Integration.

4.1 The effect on profits and welfare of horizontal integration.

There are two lines of research: one dealing with the private incentives of mergers and the other one studies their impact on welfare. This second line is very relevant, because mergers have to be approved by competition policy authorities and it gives theoretical foundation to the decisions they take. We are going to start by analyzing the profitability of mergers.

Surprisingly, the evidence on the profitability of mergers is not clear. Many studies find that mergers reduce profits. The usual explanation is that management is separated from ownership in large corporations and that managers engage in unprofitable mergers because their priority is the growth of the firms and not profits.

4.1.1 Mergers in Bertrand and Cournot competition.

The overall effect of a merger on profitability can be understood as the sum of two different effects: the first one comes from the fact that merging firms change their strategies and the second one from the fact that nomerging firms change their strategies. The first effect can only be positive because after the merger firms are maximizing joint profits while they were maximizing individual profits before the merger. This change can only increase joint profits. The sign of the effect coming from the change in the strategy of nonmerging firms is uncertain and depends crucially on whether strategies are strategic complements (Bertrand) or strategic substitutes (Cournot).

In Bertrand, merging firms increase their price after merger to maximize joint profits. For example, suppose that firm i and j merge and they sell differentiated goods.

$$\begin{aligned} \pi &= (p_i - c_i)D_i(p) + (p_j - c_j)D_j(p) \\ \frac{\partial \pi}{\partial p_i} &= D_i + (p_i - c_i)\frac{\partial D_i(p)}{\partial p_i} + (p_j - c_j)\frac{\partial D_j(p)}{\partial p_i} = 0 \end{aligned}$$

The third term would not appear if firm i had not merged with firm j. It is positive and pushes the optimal price upwards.

As we have strategic complements (best responses are upward sloping), nomerging firms react by increasing their prices as well. This increase benefits merging firms because it increases the demand for their goods. Then merger increases profits because the changes in the prices of both merging and nonmerging firms go in the direction to increase profits.

In Cournot, merging firms reduce their output to maximize joint profits. For example, suppose that firm i and j merge and they sell differentiated goods.

$$\begin{aligned} \pi &= q_i(P_i(q) - c_i) + q_j(P_j(q) - c_j) \\ \frac{\partial \pi}{\partial q_i} &= P_i(q) - c_i + q_i \frac{\partial P_i(p)}{\partial q_i} + q_j \frac{\partial P_j(p)}{\partial q_i} = 0 \end{aligned}$$

The third term would not appear if firm i had not merged with firm j. It is negative and pushes the optimal quantity downwards.

Then as we have strategic substitutes nomerging firms will react by increasing their output. This reduces the profits of merging firms because it reduces the price at which they sell their goods. Then the effect of a merger on profits is uncertain because it is the sum of a positive and a negative effect. The negative effect will be small if most firms merge because then the number of nonmerging firms will be small and their output reaction will also be small. With linear demand and costs and symmetric firms, if less than the 80% of the firms merge, the merger is not profitable. This means that the negative effect is very strong. In Figure 1 we plot for every possible number of firms in the market, the minimal market share for a merger to be profitable. The function has a minimum of 80% in $n = 5$. This is the result we have just stated.

Given the surprising result obtained with Cournot competition, a great deal of papers have tried to increase the profitability of mergers by changing the assumptions of the original model. As the problem hinges on the fact that nomerging firms react by increasing their output the first set of papers have introduced new assunptions in order to reduce the reaction of nonmerging firms. The second group of papers increases the profitability of merging firms by allowing that some efficiency gains are obtained from it. The third group explicitly considers the separation between ownership and management.

Reducing the reaction of outsiders *Convex Costs*

Instead of linear costs, Perry and Porter (1985) consider quadratic costs $C(q) = dq^2$. The greater d the flatter the reaction curve and therefore lower the reaction of nonmerging firms and more profitable mergers will be. Observe that the slope of the reaction function is given by:

$$r_i'(q_{-i}) = -\frac{1}{2(1+d)}$$

This result can be reinterpreted in terms of product differentiation because if profits are rewritten appropriately d stands as the degree of product differentiation. The profits in the Perry and Porter (1985) model are written as:

$$(\alpha - q_i - q_{-i})q_i - dq_i^2 = (\alpha - (1+d)q_i - q_{-i})q_i$$

The last expression is simply the profit of a firm selling a differentiated product with no cost. Then, we have that the greater the product differentiation the higher the profitability of mergers (Lommerud and Sorgard (1997)).

Convex demands

If Perry and Porter changed the costs from the original model, Faulí-Oller changes the shape of demand to take the following form: $P(Q) = A - \frac{1}{b+1}Q^{b+1}$. b parametrizes the degree of concavity of demand. The lower b the more convex is demand. The lower b the flatter the reaction function of firms and the more profitable mergers will be. Observe that the slope of the reaction function is given by:

$$r_i'(q_{-i}) = -1/\left(\frac{1}{1+bs_i} + 1\right)$$

If b increases, the denominator increases and the slope in absolute value also increases.

One can check that in Figure 2. It plots the minimal market share for a merger to be profitable for the case of demands of unit elasticity. It can be easily seen that the minimal market share decreases and therefore there are mergers that were not profitable in the linear case that are now profitable once we consider a more convex demand.

Sequential mergers

Mergers in Salant et. al (1983) were not profitable, because nonmerging firms react by increasing their output. This negative effect will be lower the lower the number of nonmerginng firms. Therefore a merger by reducing this number may induce new profitable mergers. Consider the following game with unit elasticity demands. Profits of firms as a function of the number of competitors n is given by $\pi(n) = \dfrac{1}{n^2}$. Firm 1 and 2 decide together whether they want to merge. Firm 3 and 4 do the same. We have the following payoff matrix. It has two equilibria either no firm merge (NM) or both firms merge (M). Then, firms want to merge if the competitor do the same. This result corresponds nicely with the empirical evidence that mergers occur in waves (Fauli-Oller (2000)).

		Firm 3&4			
		NM		M	
Firm 1&2	NM	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{9}$	$\frac{1}{9}$
	M	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{4}$	$\frac{1}{4}$

Introducing efficiency gains. So far, the only effect of mergers is to reduce competition. We have seen that, in general, this effect is too weak to induce profitable mergers. In the next model, Fauli-Oller (2002), mergers will allow to reduce costs. When this reduction is big enough, the merger will be profitable.

Assume that firms have constant marginal costs, but they may be asymmetric. When two firms with different costs merge, the merged entity produces at the cost of the efficient firm. We can imagine that firms have excess capacity and they allocate all the production in the low-cost plant.

With linear demand the merger of two firms (i is the efficient firm and j the inefficient) is profitable if:

$$\frac{s_i}{s_j} > \frac{n(n+2)}{2(n+1)} \tag{26}$$

where s_i denotes market share and n the number of firms. Using the FOC we can write the left hand side of the previous inequality as:

$$\frac{P - c_i}{P - c_j} = \frac{s_i}{s_j}$$

We can see that market share and costs are negatively correlated. Therefore, the merger will be profitable if the cost differential between merging partners is high enough. In other words, if the efficiency effect is high enough. Observe that as the r.h.s. of (26) is greater than 1, the merger of firms with the same cost is not profitable.

If demand decreases, price decreases and the previous ratio increases. Then it is more likely that the profitability condition be satisfied (observe that the right hand side of (26) remains constant). Then we can say that reductions in demand increase the profitability of mergers. There exists a long controversy on whether mergers occur in booms or in crisis. The model seems to explain the evidence that suggests that mergers happen in periods of declining demand as a way to rationalize production. The efficiency gains in the model are obtained through reallocating the production inside the merging firm.

Considering the separation between ownership and management. González-Maestre and López-Cuñat (2000) make the same assumptions as Salant et al (1983) in terms of costs, demand and type of competition, but they assume that owners delegate output decisions to managers whose incentive package is like in Ferstman and Judd (1987). They obtain that mergers become more profitable.

Finally Fauli-Oller and Motta (1996) allow managers to take also the decision to buy rivals. They obtain that some of the takeovers are not profitable, coherent with the empirical evidence we presented at the beginning.

4.2 The model of Farrell and Shapiro.

4.2.1 Motivation: 1

Once the market shares have been obtained, the premerger and the postmerger Herfindahl-Hirshman Index (HHI) can be calculated. The postmerger HHI is obtained by considering that, after the merger, the merging firms' market share will be the sum of their premerger shares. Using the postmerger HHI (HHI^p) and the increase in the HHI (ΔHHI), the Merger Guidelines suggest that a merger should not be challenged if one of the following conditions holds:

a) $HHI^p < 1000$.

b) $1000 < HHI^p < 1800$ and $\Delta HHI < 100$.

c) $HHI^p > 1800$ and $\Delta HHI < 50$.

A merger raises less antitrust concerns when either occurs in an unconcentrated market or is small (observe that in the case of two-firm mergers $\Delta HHI = 2s_i s_j$). We will check if both ideas are corroborated in the following model.

Assume that the assumptions on demand and costs we assumed while presenting the Cournot model still hold.

4.2.2 Preliminary1: The cumulative reaction function.

$$
\begin{aligned}
q_i &= r_i(Q - q_i) \\
\frac{dq_i}{dQ} &= \phi_i'(Q) = \frac{r_i'(q_{-i})}{1 + r_i'(q_{-i})} = -\lambda_i = \frac{P'(Q) + q_i P"(Q)}{C_i^{"}(q_i) - P'(Q)} < 0 \\
dq_i &= -\lambda_i dQ
\end{aligned}
$$

It measures how does a firm optimally adjust its production when total output increases. It can be calculated using the expression of the slope of the reaction function we derived in class.

4.2.3 Preliminary2: The reaction to an exogenous decrease in the prodution of a firm.

A merger is a reduction in the number of independent firms. But Farrell and Shapiro (1990) looks at the situation differently. We know that (unless costs efficiencies are very high) the merging firms will reduce their output after merger. So merger will be studied as a differentiable process where the merging firms reduce their output from the premerger level to the postmerger level. In this sense, a merger can be seen as the opposite process than entry where an entrant increased its output. Then it will have the opposite effects: the output of competitors will increase but price will also increase.

We are going to study the effect of mergers through studying how does the "exogenous" reduction in the output of merging firms affect welfare. As a first step, we are going to study how does an infinitesimal decrease in merging firms' output affect welfare. The total effect will be obtained as the sum of infinitesimal changes. We call insiders (outsiders) the firms that do (not) participate in the merger and they are denoted respectively by I and O. We do not make any spetial assumption about the evolution of the costs of insiders. Total welfare is given by:

$$
\begin{aligned}
W &= \int_0^{Q_I + \sum_{i \in O} q_i} P(x)dx - C^I - \sum_{i \in O} C_i(q_i) \\
dW &= P(dQ_I + \sum_{i \in O} dq_i) - dC^I - \sum_{i \in O} C_i'(q_i)dq_i \\
dW &= PdQ_I - dC^I + \sum_{i \in O} [P - C_i'(q_i)]dq_i
\end{aligned}
$$

The impact on welfare of an increase in the output of firm i is measured by the difference between the valuation of consumers of the marginal unit (price) and its marginal cost.

Given that $dq_i = -\lambda_i dQ$, from FOC $P - C_i'(q_i) = -q_i P'(Q)$ (big firms have big price margins), we add and substract $Q_I dP$ and $Q_I dP = Q_I P'(Q)dQ$ we have:

$$
dW = [PdQ_I - dC^I + Q_I dP] - Q_I P'(Q)dQ + \sum_{i \in O} q_i P'(Q)\lambda_i dQ
$$

To avoid having to calculate the efficiency gains of the merger that are included in dC^I, we evaluate the external effect of a merger: the effect on outsiders and consumers. Notice that $\pi^I = PQ_I - C^I <-> d\pi^I = dPQ_I + PdQ_I - dC^I$. Then,

$$dW - d\pi^I = -Q_I P'(Q)dQ + \sum_{i \in O} q_i P'(Q)\lambda_i dQ$$

$$dW - d\pi^I = (\sum_{i \in O} \lambda_i s_i - s_I)P'(Q)QdQ \tag{27}$$

The external effect of an infinitesimal reduction on the output of insiders is positive if:

$$\eta \equiv \sum_{i \in O} \lambda_i s_i - s_I \geq 0$$

given that $dQ < 0$, because a merger reduces total output.

We are interested in calculating the overall effect of a merger. Then we calculate it through adding the infinitesimal effects. Q_I^I is the (initial) output of merging firms before the merger and Q_I^F the (final) output of merging firms after the merger. We have that $Q_I^I > Q_I^F$. So the external effect of the merger can be written as:

$$\Delta W - \Delta\pi^I = \int_{Q_I^I}^{Q_I^F} \left(\frac{dW}{dQ_I} - \frac{d\pi^I}{dQ_I} \right) dQ_I$$

where for each Q_I, the integrand is evaluated assuming a Cournot equilibrium among outsiders given Q_I.

Using (27) we have:

$$\Delta W - \Delta\pi^I = \int_{Q_I^I}^{Q_I^F} \eta(Q_I)P'(Q)Q\frac{dQ}{dQ_I}dQ_I$$

Given that we are considering output reducing mergers $dQ_I < 0$, if $\eta(Q_I) \geq 0$ the previous expression is positive. A sufficient condition for this is that $\eta(Q_I^I) \geq 0$ and $\eta'(Q_I) \leq 0$. The following Proposition finds sufficient conditions for this to hold and states the result.

Proposition 2 *If*

$$P" \geq 0, P''' \geq 0, C_i'' \geq 0,\ C_i^{'''} \leq 0 \text{ that guarantee } \eta'(Q_I) < 0$$

$$\text{and } \sum_{i \in O} \lambda_i s_i^I - s_I^I \geq 0 \text{ that guarantee } \eta(Q_I^I) \geq 0$$

then an output reducing profitable merger raises welfare.

MESSAGE: ONE SHOULD APPROVE MERGERS OF SMALL FIRMS. It coincides with the spirit in the Merger Guidelines that allows mergers when they increase the Herfindahl Index slightly.

Observe that the proposition takes as given that the merger is profitable. The idea is that the Proposition should guide the decisions of antitrust authorities on whether to approve a merger. The antitrust does not evaluate all mergers but only the ones where merging firms are willing to merge and present the case for approval. If merging firms are willing to merge it is not hard to understand that it is because it is profitable. The antitrust authority takes profitability for granted and it has only to evaluate the external effect (the effect on consumers and outsiders) with the help of the previous Proposition.

There are two elements that make more likely that the external effect is positive: the size of outsiders and their reaction.

The greater the size of outsiders the more likely that the external effect is positive because their output expansion after merger have a more positive effect on welfare because they have higher price-cost margins. It is precisely the need of having big outsiders that explains that one should only approve the merger of small firms.

The greater the reaction of outsiders the more likely is that the external effect is positive. Observe that if the reaction was so important that price remained constant after merger, the external effect would be positive, because consumer surplus does not change and profits of outsiders increase (observe that after merger they still maximize and the output of the other firms is lower, because they produce more). On the other hand, if outsiders "did not respond ($\lambda_i = 0$) then every output reduction would be bad for rivals and consumers jointly: rivals would benefit, but consumers would lose for more".

Recall what we saw in the profitability section that the reaction of ousiders reduces merger profitability. Then although an increase in the reaction of outsiders makes easier that the condition holds, it makes stronger the other assumption that mergers are profitable in the sense that more cost efficiencies are needed to be satisfied.

4.2.4 Application to the Salant et al. model.

In this case of linear demand and costs and symmetric firms, we have that $\lambda_i = 1$. Then the condition is $\sum_{i \in O} s_i - s_I \geq 0$ (the market share of outsiders is greater than the one of insiders). Given that market shares are symmetric and if we consider that $m+1$ firms merge the result can be stated as:

$$\begin{aligned} \frac{n-m-1}{n} - \frac{m+1}{n} &\geq 0 \\ \frac{n}{2} &\geq m+1 \end{aligned}$$

How can we reconcile the result with the fact that in this setting, welfare is increasing in the number of firms ? We know that, without further assumptions, mergers are not profitable and therefore merger profitability requires that some savings in fixed costs are obtained. If this is the case, the fact that welfare may increase comes as no surprise.

4.2.5 Application to the Perry and Porter(1985) and McAfee and Williams (1988) model.

$$P = A - Q \text{ and } C(q_i) = d_i q_i^2.$$

From the FOC we have $P = +q_i + C_i'(q_i)$ and given that marginal costs are linear we have that $C_i'(q_i) = C_i^{"}(q_i)q_i$. Then $\lambda_i = -\frac{-1}{\frac{C_i'(q_i)}{q_i} + 1} = \frac{q_i}{P}$. Given that demand is linear, the elasticity reads: $\frac{P}{Q}$ and therefore $\lambda_i = \frac{s_i}{\varepsilon}$. In this case, it will be good to have big outsiders, not only because their output increases have a greater positive impact on welfare, but because they simply react more. Those two effects are reflected in the final result:

$$s_I < \frac{1}{\varepsilon} \sum_{i \in O} s_i^2$$

The greater the concentration the more likely that the external effect is positive. This goes against the presumption that concentration is bad in terms of the welfare effects of mergers.

4.3 Sequential monopolies.

Assume that firm U produces an input at cost c. A downstream firm D transforms the input into a final good on a one-for-one basis and at zero marginal cost (one cant think that firm D is simply a retailer). Market demand is given by $P = \alpha - Q$. We are going to analyze a game where in the first stage firm U sets the price for the input (w). Then firm D decides how many units of the input to buy and after transforming them into a final good he sells them to consumers.

w determines the marginal cost of firm D and we can calculate the amount he wants to sell at this cost:

$$Q = \frac{\alpha - w}{2}$$

This determines the units of the input he wants to buy to firm U. Then the profit of firm U is given by:

$$(\frac{\alpha - w}{2})(w - c)$$

It is maximized at:

$$w^* = \frac{\alpha + c}{2}$$

The profits of firm U and firm D are given respectively by $\frac{1}{8}(\alpha - c)^2$ and $\frac{1}{16}(\alpha - c)^2$. Market price is given by $\frac{3\alpha + c}{4}$. One can check that the profits of the industry are lower and price higher than if firm U and D were vertically integrated. This is the result of having "double marginalization". Under vertical separation, there are two monopoly pricing decisions being made. If the only contractual instrument that firm U and firm D can use is the wholesale price, then two monopolies margins will be added to marginal cost, resulting in a price greater than monopoly price. In the problem set you are asked to see that the problem disappears when the downstream sector is competitive and charges no margin.

Now we are going to see that we can replicate the situation with vertical integration if firm U can use two-part tariff supply contracts including a per unit price w and a fixed part F.

We proceed to derive the optimal behavior of firm U when receiving a two-part tariff contract $wq + F$. If he decides to buy the input, he will buy $\frac{a-w}{2}$ units and will obtain profits of $\left(\frac{a-w}{2}\right)^2 - F$. Buying input is optimal if the previous expression is positive.

Firm U, when choosing the optimal contract, will maximize:

$$\begin{aligned} & \underset{w,F}{Max}(w-c)\left(\frac{a-w}{2}\right)+F \\ s.t.0 \quad \leq & \quad \left(\frac{a-w}{2}\right)^2 - F \end{aligned}$$

As the restriction is alwasys binding, the maximization program can be rewritten as:

$$\underset{w}{Max}(w-c)\left(\frac{a-w}{2}\right)+\left(\frac{a-w}{2}\right)^2$$

The F.O.C. of the maximization proces is given by:

$$\begin{aligned} a-w-w+c-a+w &= 0 \\ w &= c \end{aligned}$$

With the fixed part, firm U extracts all the rents $F = \left(\frac{a-c}{2}\right)^2$. We have the same situation as with vertical integration.

4.4 Competition downstream (observable contracts).

We are going to introduce competition downstream. The question is then whether the monopolization of one stage of production implies the monopolization of the whole industry. Assume we have n firms downstream.We analyze the following game.

In the first stage, firm U offers supply two-part tariff contracts to downstream firms. Contracts may be different for each firm $(w_i q_i + F_i)$.

In the second stage, downstream firms decide whether to accept the supply contracts. Accepting the contract means paying the fixed part F_i.

In the third stage sales are made.

If each firm has accepted a contract $w_i q_i + F_i$, the individual production will be given by $q_i = \dfrac{\alpha - nw_i + \sum_{j\neq i} w_j}{n+1}$, total output by $Q = \dfrac{n\alpha - \sum_{i=1}^{n} w_i}{n+1}$ and individual profits by $\pi_i = q_i^2$. Firm U will extract all the profits obtained by downstream firm through the fixed part and therefore its profits will be given by:

$$
\begin{aligned}
\Pi = & \sum_{i=1}^{n}(w_i - c)q_i + \sum_{i=1}^{n}\pi_i \\
\Pi = & \sum_{i=1}^{n}\left(\frac{\alpha - nw_i + \sum_{j\neq i} w_j}{n+1}\right)\left((w_i - c) + \frac{\alpha - nw_i + \sum_{j\neq i} w_j}{n+1}\right) \\
\Pi = & \sum_{i=1}^{n}\left(\frac{\alpha - nw_i + \sum_{j\neq i} w_j}{n+1}\right)\left(\frac{\alpha - (n+1)c + \sum_{i=1}^{n} w_i}{n+1}\right) \\
\Pi = & \left(\frac{\alpha - (n+1)c + \sum_{i=1}^{n} w_i}{n+1}\right)\left(\frac{n\alpha - n\sum_{i=1}^{n} w_i + (n-1)\sum_{i=1}^{n} w_i}{n+1}\right) \\
\Pi = & \left(\frac{\alpha + \sum_{i=1}^{n} w_i - (n+1)c}{n+1}\right)\left(\frac{n\alpha - \sum_{i=1}^{n} w_i}{n+1}\right)
\end{aligned}
$$

The FOC of the maximization process are given by:

$$\frac{\partial \Pi}{\partial w_i} = n\alpha - \sum_{i=1}^{n} w_i - \alpha - \sum_{i=1}^{n} w_i + (n+1)c = 0$$

The optimal contracts satisfy:

$$\sum_{i=1}^{n} w_i = \frac{(n-1)\alpha + (n+1)c}{2} = n\left(\frac{\alpha + c}{2} - \frac{\alpha - c}{2n}\right)$$

The process does not determine the individual values of w_i but only its sum. The important thing is that the sum of wholesale prices makes that total output equals the monopoly output:

$$\frac{n\alpha - \sum_{i=1}^{n} w_i}{n+1} = \frac{2n\alpha - (n-1)\alpha - (n+1)c}{2(n+1)} = \frac{\alpha - c}{2}$$

and the upstream firm obtains the whole monopoly profits. (It includes as a particular case, the case where only one firm produces. It receives a wholesale price equal to marginal cost and the other firms receive a wholesale price equal to the monopoly price such that they do not want to produce $\frac{\alpha + c}{2}$) In this case, we have that the monopolization of one part of the business (in this case, the production of the input) implies the monopolization of the whole business.

4.5 Competition downstream (unobservable contracts).

We are going to see if this situation can be sustained in equilibrium when the supply contracts are not observable or secret. As contracts are not observed we have a game of incomplete information. We will have a multiplicity of perfect bayesian equilibria depending on the beliefs of the downstream firms about the contracts offered to competitors when they are offered an off-the-equilibrium contract. In equilibrium we know that downstream firms must have correct beliefs about the contracts received by others, but bayesian equilibrium does not impose any restriction on off-the-equilibrium beliefs. The literature analyzes two types of beliefs about what happens when one is offered an off-the-equilibrium contract.

-Symmetric beliefs. The downstream firm believes that the others also receive the same contract.

-Passive beliefs: The downstream firm believe that the others still receive the equilibrium contracts.

We are going to see if the contracts that replicated the monopoly situation are equilibrium contracts in the two previous cases. We focus on the case of symmteric contracts ($w = \frac{(n-1)\alpha+(n+1)c}{2n} = \frac{\alpha+c}{2} - \frac{\alpha-c}{2n}$ and each firm produces $\frac{\alpha-c}{2n}$).

4.5.1 Symmetric beliefs.

They will not be equilibrium contracts if firm U can change the contracts and increase its profits. We are going to check if firm U can gain by changing the contract of one downstream firm. As the others will not observe the change, they will not change their behavior and what firm U obtains from them will not be affected. Therefore we should focus on the change of revenues obtained from the downstream firm whose contract has been changed. If firm U sets w_i, the downstream firm will think that the others also receive w_i, and therefore he will produce $\frac{\alpha-w_i}{n+1}$ and will obtain profits of $\left(\frac{\alpha-w_i}{n+1}\right)^2$. Therefore, firm U will obtain from the downstream firm:

$$\begin{aligned}
I &= (w_i - c)\left(\frac{\alpha-w_i}{n+1}\right) + \left(\frac{\alpha-w_i}{n+1}\right)^2 \\
\frac{\partial I}{\partial w_i} &= \left(\frac{1}{n+1}\right)^2 ((\alpha - w_i - w_i + c)(n+1) - 2(\alpha - w_i)) = 0 \\
w_i &= \frac{(n-1)\alpha+(n+1)c}{2} = \frac{\alpha+c}{2} - \frac{\alpha-c}{2n}
\end{aligned}$$

He wants to set the equilibrium contract. Firm U can not gain by changing the contract and therefore the contracts that sustain the monopoly outcome are equilibrium contracts.

4.5.2 Passive beliefs.

We are going to see that with passive beliefs, those contracts are not equilibnrium contracts, because firm U can obtain more profits by setting a different contract to a downstream firm i. When firm U changes the contract to downstream firm i, the other downstream firms do not realize it, they will not change their behavior and firm U will obtain from them the same revenue. Therefore, to see if changing the contract is profitable we have to see the evolution of the revenues obtained from firm i. If firm i is offered a contract with w_i, firm i will think that the other downstream firm recieve the equilibrium contracts and they are still producing the same amount. Therefore, he will produce:

$$q_i = \frac{\alpha - (n-1)\dfrac{\alpha - c}{2n} - w_i}{2}$$

He will obtain profits of $\pi_i = q_i^2$. Firm U will set a fixed part equal to this amount, obtaining from firm i the following revenues:

$$\begin{aligned}
I &= \left(\frac{\alpha - (n-1)\dfrac{\alpha - c}{2n} - w_i}{2}\right)\left((w_i - c) + \frac{\alpha - (n-1)\dfrac{\alpha - c}{2n} - w_i}{2}\right) \\
I &= \left(\frac{\alpha - (n-1)\dfrac{\alpha - c}{2n} - w_i}{2}\right)\left(\frac{\alpha - (n-1)\dfrac{\alpha - c}{2n} + w_i - 2c}{2}\right) \\
\frac{\partial I}{\partial w_i} &= -\alpha + (n-1)\frac{\alpha - c}{2n} - w_i + 2c + \alpha - (n-1)\frac{\alpha - c}{2n} - w_i = 2c - 2w_i = 0 \\
w_i &= c
\end{aligned}$$

It is different than the equilibrium contract. Then, firm U by changing the contract can increase its profits. One can check that with the fixed part obtains more than the nth part of the monopoly profits.

$$F = \left(\frac{a - c - \frac{(n-1)(a-c)}{2n}}{2}\right)^2 = \left(\frac{(a-c)(n+1)}{4n}\right)^2 > \frac{1}{n}\left(\frac{a-c}{2}\right)^2$$

The contract we have obtained is like the one we had found for the case of sequential monopolies. With unobservable contracts and passive beliefs is like if the downstream firm i is a monopolist in the residual demand $P = \alpha - \sum_{j \neq i} q_j - Q$, where $\sum_{j \neq i} q_j$ is the production of the other downstream firms (in equilibrium).

Which is the equilibrium with unobservable contracts and passive beliefs ?

With a similar reasoning, one can see that contracts such that $w_i \neq c$ can not be equilibrium contracts, because firm U would increase its profits from i by setting $w_i = c$, given that the others do not realize it. Therefore, in equilibrium we must have that for all i $w_i = c$. In equilibrium downstream firm have correct beliefs about equilibrium contracts and therefore they will produce the Cournot output with n symmteric firms with cost c. With the fixed part, firm U extracts all the rents and therefore the fixed part amounts to the profits with Cournot competition $F = \left(\dfrac{a-c}{n+1}\right)^2$.

The number of downstream firms positively affects the degree of competition in the market. This result shows that liberalizing parts of an industry, like it has been done in telecommunications and electricity, may have a lot of sense although some other parts of the industry are still monopolized for technological reasons.

How can firm U achieve the monopoly situation ? Vertically integrating with one downstream firm. In this case, all the sales (amounting to the monopoly output) will be done by the integrated firm. The integrated firm will not supply the other downstream firms, because it would decrease its profits. We have found a reason to explain vertical mergers that we did not have with observable contracts where the monopoly solution was obtained without the need of vertical integration.

REMARK: Sometimes, one forbids to discriminate between customers i.e. to supply the same good at different prices. It may be reasonable in other contexts, but in the setting we are analyzing it will help to sustain the monopoly solution even with unobservable contracts. The reason is that it makes that downstream firm have symmetric beliefs that we know that they sustain the monopoly situation: if one is offered an off-the-equilibrium contract, he knows that legally firm U is obliged to offer the same contract to the other downstream firms.

Before ending chapter IV, one final comment connecting the situation in markets with upstream and downstream firms with our initial intend to explain horizontal mergers. We saw that in the standard setting mergers failed in many cases to be profitable. How would things change if (downstream) firms have to buy the input from a supplier? Bru and Faulí-Oller (2003) analyze precisely that in the context of Rey and Tirole model with passive conjectures. For downstream firms to have positive profits, they assume that there is an alternative supplier of the input at cost $\overline{c}$. Then profits of downstream firms are given by (you are asked to obtain that in an exercise):

$$\begin{aligned} & (\alpha - c)\left(\frac{1}{n+1} - \frac{\theta}{2}\right)^2 \\ \text{where } \theta &= \frac{\overline{c} - c}{\alpha - c} \end{aligned}$$

Observe that when $\theta = 0$, we obtain the Cournot profits. The result is that any merger is profitable if θ is high enough. If θ is high the upstream firm has a lot of selling power and then it is when downstream badly need to merge to countervail this power.

5 Product differentiation.

Goods can be differentiated in two different dimensions. Horizontal differentiation appears because consumers like variety whereas vertical differentiation arises from the fact that consumers appreciate quality. Shirts of different colour or design are horizontally differentiated while computers with different microprocessors are vertically differentiated.

Vertical differentiation refers to the idea that it is possible to rank the different goods from high to low quality. All consumers agree on this ranking[4]. On the contrary, horizontal differentiation refers to the idea that the preferences of consumers over the goods are heterogeneous. Some people prefer blue shirts whereas others prefer red shirts.

5.1 Model of product differentiation.

5.1.1 Choice of varieties.

Now we are going to present a model of horizontal differentiation. To represent our market we are going to use a segment of length 1 (the famous Hotelling segment). Each point in the segment represents the location of a consumer and we assume that consumers are uniformly distributed in the segment.

Assume that 2 firms (A and B) can simultaneously locate a shop of the same good in a point of the segment. For the time being we assume that the price of the good is regulated at a level greater than marginal cost. In this case, the only strategic variable of a firm is its location. Every consumer wants to buy one and only one unit of the good. Consumers when choosing where to buy, choose the shop closest to their location. This simple behavioral rule has the following implication as far as the demand each firm obtains.

If firm A locates in a and firm B locates in b ($a < b$) all consumers to the left of $\frac{b+a}{2}$ will buy from A and the rest of consumers will buy from B. The consumer in $\frac{b+a}{2}$ is at the same distance from a than from b, then he is indifferent between buying from firm A and from firm B. That is why he is known as the "indifferent consumer". If both firms locate in the same point, we assume that they share equally demand.

We are going to analyze the equilibrium of the location game. As price is fixed at a level greater than marginal cost, maximizing profits is equivalent to maximizing demand. This implies that in equilibrium, firms should locate in the same point. Otherwise a firm could increase its demand by approaching its location to the one of the competitor. Symmetric locations in a point different that 1/2 are not an equilibrium because a firm can increase its demand by locating in 1/2. Then in equilibrium both firms locate in 1/2.

So far we have related the points in the segment with geograhic locations. Another interpretation, that broadens the scope of the analysis, consists in identifying the different points as possible varieties of a good (e.g. the degree of

[4]This does not mean that all consumers buy the same good if they are offered at different prices.

sweetness of a chocolate). Then, the location of a consumer refers to the variety he prefers. He will buy from the existing variety closest to his ideal point.

This model has been used to analyze questions on political party competition when political platforms are characterized by the value of a parameter (e.g. the level of taxation). Consumers become voters and the same logic applies. The result of the model (both parties would choose the same platform) has been used to explain the convergence of platforms in a democracy. One should note that the result changes with more than two parties.

5.1.2 Choice of prices.

Now we are going to analyze the situation where locations are given and firms choose prices. If prices are different, consumers, before deciding where to buy the good, have to take into account two different things:

- the distance between its location and the location of firms.
- the price firms charge.

To be able to aggregate both elements, we assume that the disutility of travelling to a shop can be represented as a transportation cost expressed in monetary terms. We are going to assume that transportation costs can be written as a quadratic function of the distance (d) to the shop $C(d) = td^2$. The greater t, the greater the cost of travelling. Consumers will buy from the shop whose sum of price and transportation cost is lower.

Assume Firm A is located at z and firm B at $1 - z$, where $z \leq 1/2$. We are going to check that the behavioral rule we have set allows us to compute the demand of both firms. In order to that, we are going to find the location (x) of a consumer that is indifferent between buying from A than from B. x satisfies:

$$p_A + t(x - z)^2 = p_B + t(1 - z - x)^2$$

$$p_A + tx^2 + tz^2 - 2tzx = p_B + t + tz^2 - 2zt + tx^2 - 2tx + 2tzx$$

$$x2t(1 - 2z) = p_B - p_A + t(1 - 2z)$$

$$x = \frac{p_B - p_A}{2t(1 - 2z)} + \frac{1}{2}$$

$$x_B = 1 - x = \frac{p_A - p_B}{2t(1 - 2z)} + \frac{1}{2}$$

The demand of firm A will be x and the demand of firm B $1 - x$.

We have that:

$$\frac{\partial x}{\partial p_B} = \frac{1}{2t(1 - 2z)} \tag{28}$$

The lower (28) the greater the product differentiation. It is decreasing in t and increasing in z.

The analysis below will show that prices and profits of firms are directly determined by the degree of product differentiation. Profits of firm $i(i = A, B, i \neq j)$ are given by:

$$\Pi_i = (p_i - c)\left(\frac{p_j - p_i}{2t(1-2z)} + \frac{1}{2}\right)$$

$$\frac{\partial \Pi_i}{\partial p_i} = \frac{p_j - p_i}{2t(1-2z)} + \frac{1}{2} - \frac{p_i - c}{2t(1-2z)} = 0 \tag{29}$$

The solution to (29) gives us the equilibrium:

$$p_A^* = p_B^* = c + t(1-2z)$$

Then equilibrium profits are given by:

$$\Pi^* = (\frac{1}{2})t(1-2z)$$

Both prices and profits are increasing in the degree of product differentiation (it increases with t and decreases with z). Observe that if product differentiation disappears (either $t = 0$ or $z = \frac{1}{2}$), the original Bertrand result obtains where price equals marginal cost.

5.1.3 Choice of varieties and prices.

In the two previous sections, we have seen the choice of location if prices were given and the choice of prices if locations were given. Now we are going to find the equilibrium if both decisions are endogeneous. We consider the following two-stage game where in the first stage firms choose location (firm A chooses location a and firm B location b, where $a < b$, withou loss of generality) and, in the second stage, they choose prices (firm A chooses price p_A and firm B price p_B).

Before solving the model, we recall two effects, that appeared separately in the two previous models, that are going to be present in this more complex model.

- *Demand effect*: the sales of a firm increase when it approaches its location to the one of the competitor.

- *Competition effect*: the price of the competitor increases when a firm moves away from it.

We are going to solve the second stage. It is more complicated that what we had previously done because the equilibrium should be obtained for all possible locations and not only for the symmetric ones. Assume that firm A has chosen location a and firm B the location b, where $a < b$. In the first place, we find the indifferent consumer x that satisfies:

$$p_A + t(x-a)^2 = p_B + t(b-x)^2$$

$$\begin{aligned} x &= \frac{p_B - p_A + t(b^2 - a^2)}{2t(b-a)} = \frac{p_B - p_A}{2t(b-a)} + \frac{b+a}{2} = \\ &= a + \frac{b-a}{2} + \frac{p_B - p_A}{2t(b-a)} \end{aligned}$$

Profits of firms are given by:

$$\Pi_A = (p_A - c)x \text{ and } \Pi_B = (p_B - c)(1-x)$$

The Nash equilibrium is obtained by solving the system of equations of the FOC.

$$\frac{\partial \Pi_A}{\partial p_A} = x - \frac{p_A - c}{2t(b-a)} = 0 \text{ and } \frac{\partial \Pi_B}{\partial p_B} = 1 - x - \frac{p_B - c}{2t(b-a)} = 0$$

Equilibrium prices are given by:

$$p_A = c + \frac{t(b-a)(2+b+a)}{3} \text{ and } p_B = c + \frac{t(b-a)(4-b-a)}{3}. \tag{30}$$

We have the competition effect because the price of the competitor increases when a firm moves away from it:

$$\frac{\partial p_A}{\partial b} = \frac{t}{3}(2+b) > 0 \text{ and } \frac{\partial p_B}{\partial a} = \frac{t}{3}(-4+2a) < 0$$

Sales in equilibrium are given by:

$$x_A = \frac{2+b+a}{6} \text{ and } x_B = \frac{4-b-a}{6} \tag{31}$$

We have the demand effect because a firm increases its sales when it moves closer to its competitor:

$$\frac{\partial x_A}{\partial a} > 0 \text{ and } \frac{\partial x_B}{\partial b} < 0$$

From (30) and (31) we can derive the equilibrium profits as a function of locations:

$$\Pi_A = \frac{t(b-a)(2+b+a)^2}{18} \text{ and } \Pi_B = \frac{t(b-a)(4-b-a)^2}{18}$$

$$\frac{\partial \Pi_A}{\partial a} = t(2+b+a)(-2+b-3a) < 0 \tag{32}$$

$$\frac{\partial \Pi_B}{\partial b} = t(4-b-a)(4-3b+a) > 0 \tag{33}$$

Firms want to move away as much as possible from the competitor. In equilibrium they will choose $a^* = 0$ and $b^* = 1$.

This result seems to show that the competition effect always dominates the demand effect. This holds if we impose that locations should be inside the segment [0,1]. If we suppress this assumption and allow that firms locate in any point over the real line, the previous derivatives may have different signs. The equilibrium in this case would be the solution of the system of equations of the first order conditions (32) and (33). We can check that it is[5]:

$$a^* = -\frac{1}{4} \text{ and } b^* = \frac{5}{4}$$

5.2 Inexistence of equilibrium in the Hotelling model with linear transportation costs.

The previous two-stage game has no equilibrium in pure-strategies with linear transportation costs. The reason is that for close enough locations, the second stage has no equilibrium in pure-strategies, because profit functions are discontinuous. The discontinuity appears in the price ($\bar{p}$)such that a firm attracts all consumers located between the firms, because then it also attracts all consumers located on the other side of the rival.

For $p > \bar{p}$ (the indiferent consumer lies between the location of both firms), the profit function is continuous and concave with a local maximum at p^*.[6]

For $p < \bar{p}$ the profit function is increasing, because demand is inelastic.

The optimal decision of a firm is either $\bar{p}$ or p^*. As in equilibrium a firm behaves optimally, it has to choose one of them. However, a situation where a firm chooses $\bar{p}$ can not be an equilibrium because the competitor does not sell and, therefore, it does not behave optimally. Therefore, in equilibrium both firms have to choose p^*. Those are the only prices that can conform an equilibrium. We are going to calculate them for the case of symmetric locations. Firm A is located at z and firm B at $1 - z$, where $z < 1/2$.

To find the demand of each firm for prices in the concave region (the indifferent consumer lies between the locations of both firms), we have to find the indifferent consumer:

$$p_A + t(x - z) = p_B + t(1 - x - z)$$

$$2tx = p_B - p_A + t$$

$$x = \frac{p_B - p_A}{2t} + \frac{1}{2}$$

This is the demand of firm A and $1 - x$ is the demand of firm B.

The profit function of firm $i(j \neq i)$ is given by

$$\Pi_i = (p_i - c)(\frac{p_j - p_i}{2t} + \frac{1}{2})$$

[5] The other two solutions $\{a = -\frac{5}{2}, b = \frac{1}{2}\}$ and $\{a = \frac{1}{2}, b = \frac{7}{2}\}$ do not constitute an equilibrium, because the firm choosing an extreme location is not maximizing.

[6] $\bar{p}$ and p^* depend on the price set by the competitor. We abstract from this relationship to allow them to generically represent respectively the price where the profit function is discontinuous and the local maximum in the concave region.

$$\frac{\partial \Pi_i}{\partial p_i} = \frac{p_j - p_i}{2t} + \frac{1}{2} - \frac{p_i - c}{2t} = 0 \quad (34)$$

$$\frac{\partial^2 \Pi_i}{(\partial p_i)^2} = -1/t$$

Indeed, in the region where the indifferent consumer lies between the location of firms, the profit function is concave.

$$p_A = p_B = p^* = c + t \quad (35)$$

satisfy (34) for $i = A, B$. Those are the only prices that can conform an equilibrium. To be an equilibrium firms have to obtain more profits than with the highest price such that they monopolize the market $\bar{p}$. With prices (35) it obtains

$$\Pi = \frac{t}{2}.$$

$\bar{p}$ is obtained by imposing that the indiferent consumer is situated where the competitor is located.

$$\begin{aligned} c + t &= \bar{p} + t(1 - 2z) \\ \bar{p} &= c + 2tz \end{aligned}$$

With this price it obtains a profit of $2tz$. (35) is an equilibrium if

$$\begin{aligned} 2tz &\leq t/2 \\ z &\leq 1/4 \end{aligned}$$

For closer locations, there is no equilibrium in pure-strategies.

5.3 The Hotelling model with price discrimination.

So far we have assumed that firms had to set the same price to all consumers. Now instead we are going to assume that firms are able to set different prices to every consumer i.e. they can perfectly price discrminate. To ease the presentation assume that transportation costs are paid by firms. Firms will compete for every consumer. The cost to supply the consumer located in x for firm A (located in a) is $c+t(x-a)^2$ while the cost for firm B (located in b) is $c+t(x-b)^2$. We will have Bertrand competition with homogeneous goods and asymmteric costs: in equilibrium the low cost firm will supply the good at a price equal to the cost of the high cost firm.

Firm A is the low cost firm for $x < \frac{a+b}{2}$. He will sell the good at a price equal to the cost of the competitor. From selling to the consumer in x, he obtains:

$$t(b-x)^2 - t(a-x)^2 = t(b^2 - a^2 - 2(b-a)x) = (b-a)t(b+a-2x).$$

Then in all locations obtains:

$$\begin{aligned}\Pi_A &= \int_0^{(a+b)/2} (b-a)t(b+a-2x)dx = (b-a)t[(b+a)x - x^2]_0^{(a+b)/2} = \\ &(t/4)(b-a)(b+a)^2\end{aligned}$$

For firm B we have:

$$\begin{aligned}\Pi_B &= \int_{(a+b)/2}^{1} (a-b)t(a+b-2x)dx = (a-b)t[(a+b)x - x^2]_{(a+b)/2}^{1} = \\ &(t/4)(b-a)(2-b-a)^2\end{aligned}$$

The equilibrium in the first stage is obtained by solving the following system:

$$\begin{aligned}\frac{\partial \Pi_A}{\partial a} &= t(b+a)(-3a+b) = 0 \\ \frac{\partial \Pi_B}{\partial b} &= t(b+a)(2+a-3b) = 0\end{aligned}$$

The equilibrium is given by:

$$a^* = \frac{1}{4} \text{ and } b^* = \frac{3}{4}$$

They are the locations that minimize transportation costs:

$$\int_0^{(a+b)/2} t(a-x)^2 dx + \int_{(a+b)/2}^{1} t(b-x)^2$$

5.4 The circular city.

We are going to consider a model of entry. To avoid that there are locations that are a priori better than another, we assume that consumers are uniformly distributed in a circle with a perimeter equal to 1 (in this way we avoid the corners). Firms are also located in the circle. Consumers have to walk over the circle to reach the firms. Cost of firms and transportation costs are like in Section 5.1.

Salop (1979) considers the following two stage game.

In the first stage, potential entrants decide whether to enter or not. If they enter they have to pay a fixed cost F. Firms that enter do not choose their location but rather they are automatically located equidistant from one another.

In the second stage, firms compete in prices.

First of all, we solve the second stage with n firms in the market along the lines we followed in Section 5.1.

We compute the profits of firm i whose neigbours are firm k and firm j. We have to find the location (x_j) of the indifferent consumer between firm i and firm j.

$$\begin{aligned} p_i + t x_j^2 &= p_j + t(\frac{1}{n} - x_j)^2 \\ x_j &= \frac{p_j - p_i + \frac{t}{n^2}}{\frac{2t}{n}} \end{aligned}$$

Similarly we can find the location of the indifferent consumer between firm i and firm k:

$$x_k = \frac{p_k - p_i + \frac{t}{n^2}}{\frac{2t}{n}}$$

Then the profits of firm i are given by:

$$\begin{aligned} \pi_i &= (p_i - c)(x_j + x_k) \\ \frac{\partial \pi_i}{\partial p_i} &= p_j - p_i + p_k - p_i + \frac{2t}{n^2} - 2p_i + 2c = 0 \end{aligned}$$

We can obtain the eqilibrium by imposing symmetry $(p_j = p_i = p_k = p)$ in the FOC:

$$p = c + \frac{t}{n^2}$$

$$\pi = \frac{t}{n^3}$$

From the zero profit condition we obtain the number of firms that enter in the market in the first stage.

$$\begin{aligned} \frac{t}{n^3} - F &= 0 \\ n &= \left(\frac{t}{F}\right)^{1/3} \end{aligned}$$

We are going to compare this level of entry with the one that would maximize the social welfare. As we have assumed that consumers buy one (and only one) unit of the good independently of the price, a price over marginal cost does not introduce any distortion. The number of firms does not affect consumption either and it only affects the transportation costs. Then the socially optimal number of firms is the one that minimizes the sum of fixed and transportation costs:

$$\begin{aligned} C &= nF + 2n \int_0^{1/2n} t x^2 dx = nF + [\frac{tx^3}{3}]_0^{1/2n} = nF + \frac{t}{12n^2} \\ \frac{\partial C}{\partial n} &= F - \frac{2t}{12n^3} = 0 \\ n &= \left(\frac{t}{6F}\right)^{1/3} \end{aligned}$$

As with Cournot competition, we have a result of excess entry.

5.5 Vertical differentiation

We start by defining the utility of a consumer when buying a good of quality q at a price p:

$$-p + \theta q$$

θ is uniformly distributed in the support $[\underline{\theta}, \bar{\theta}]$. Qualities can take values in $[\underline{q}, \bar{q}]$. At the same price all consumers agree to prefer the good of highest quality. That is why this is a model of vertical differentiation. We assume that $\underline{\theta}$ is high enough such that all consumers want to buy one unit of the good (the market is said to be covered). There are no production costs.

We are going to solve the same two-stage game as in Section 5.1.3. We have two firms: firm 1 and firm 2. In the first stage, they choose qualities and in the second stage they choose prices. In the first place, we are going to solve the model for the case $2\underline{\theta} < \bar{\theta}$ (preferences of consumers are very heterogeneous).

In the second stage, given qualities $q_1 < q_2$, we have to calculate the equilibrium in prices. To compute the demand of each firm we have to find the indifferent consumer between low and high quality goods.

$$\begin{aligned} -p_1 + \theta q_1 &= -p_2 + \theta q_2 \\ \theta &= \frac{p_2 - p_1}{q_2 - q_1} \end{aligned} \tag{36}$$

The profit function of the low and high quality firms are given respectively by:

$$\begin{aligned} \pi_1 &= \left(\frac{p_2 - p_1}{q_2 - q_1} - \underline{\theta} \right) p_1 \\ \pi_2 &= \left(\bar{\theta} - \frac{p_2 - p_1}{q_2 - q_1} \right) p_2 \end{aligned}$$

$$\frac{\partial \pi_1}{\partial p_1} = p_2 - 2p_1 - \underline{\theta}(q_2 - q_1) = 0$$

$$\frac{\partial \pi_2}{\partial p_2} = \bar{\theta}(q_2 - q_1) + p_1 - 2p_2 = 0 \tag{37}$$

The solution to the system gives the following equilibrium prices and quantities.

$$\begin{aligned} p_1 &= \left(\frac{\bar{\theta} - 2\underline{\theta}}{3} \right) (q_2 - q_1) \text{ and } p_2 = \left(\frac{2\bar{\theta} - \underline{\theta}}{3} \right) (q_2 - q_1) \\ x_1 &= \left(\frac{\bar{\theta} - 2\underline{\theta}}{3} \right) \text{ and } x_2 = \left(\frac{2\bar{\theta} - \underline{\theta}}{3} \right) \end{aligned}$$

Then we have that prices are increasing in the level of product differentiation (competition effect) whereas quantities are constant and do no depend on the quality differential (no demand effect). (36) shows that, given prices, demand increases when the low quality firm increases its quality. But prices do adjust and the adjustment is such that sales of the low quality firm do not depend on the quality he chooses. In this case, it is easy to predict that firms will choose maximal differentiation. This is apparent when we compute the profits of firms as a function of qualities:

$$\pi_1 = \left(\frac{\bar{\theta} - 2\underline{\theta}}{3}\right)^2 (q_2 - q_1) \text{ and } \pi_2 = \left(\frac{2\bar{\theta} - \underline{\theta}}{3}\right)^2 (q_2 - q_1)$$

Then in equilibrium we have maximal differentiation:

$$q_1^* = \underline{q} \text{ and } q_2^* = \bar{q}$$

Then to reduce the intensity of competition a firm prefers to deteriorate the quality it offers even without cost savings.

We proceed to solve the case with small heterogeneity in the preferences $2\underline{\theta} > \bar{\theta}$. In the previous expressions this would mean that the low quality firm sets a negative price. As this can not be, he will set a price of zero and the optimal response of firm 2 to this price is given by (37) $p_2 = \dfrac{\bar{\theta}(q_2 - q_1)}{2}$. At this price even the consumer with lowest taste for quality prefers to buy the high quality good, so that firm 1 does not sell.

$$\begin{aligned} -\frac{\bar{\theta}(q_2 - q_1)}{2} + \underline{\theta} q_2 &> \underline{\theta} q_1 \\ (2\underline{\theta} - \bar{\theta})(q_2 - q_1) &> 0 \end{aligned}$$

In this case, even if the cost of entry was very small, we would have that only one firm would enter in this market, because a second firm can not make positive profits. No we are going to see how does this result generalize to n firms.

5.5.1 The case of n firms.

We have n firms. Each one produces a good of different quality q_i. Without loss of generality, assign to every firm a natural numer from 1 to n according to its quality $q_1 < q_2.... < q_{n-1} < q_n$. As in the previous case with two firms one has to find the indiferent consumer between two neighboring qualities.

$$\begin{aligned} -p_{i+1} + \ddot{\theta}_i q_{i+1} &= -p_i + \ddot{\theta}_i q_i \\ \ddot{\theta}_i &= \frac{p_{i+1} - p_i}{q_{i+1} - q_i} \\ for\ i &= 1...n - 1 \end{aligned}$$

We assume that the market is covered. Then, the profits of firms can be written as:

$$\begin{aligned}
\pi_1 &= p_1(\ddot{\theta}_1 - \underline{\theta}) \\
\pi_i &= p_i(\ddot{\theta}_i - \ddot{\theta}_{i-1}) \text{ for } 1 < i < n \\
\pi_n &= p_n(\bar{\theta} - \ddot{\theta}_{n-1})
\end{aligned}$$

If an equilibrium exists such that all firms sell a positive quantity, the following system of FOC should hold for the equilibrium prices:

$$\begin{aligned}
x_1 &= \ddot{\theta}_1 - \underline{\theta} = \frac{p_1}{q_2 - q_1} \\
x_i &= \ddot{\theta}_i - \ddot{\theta}_{i-1} = \frac{p_i}{q_{i+1} - q_i} + \frac{p_i}{q_i - q_{i-1}} \text{ per } i = 2...n-1 \\
x_n &= \bar{\theta} - \ddot{\theta}_{n-1} = \frac{p_n}{q_n - q_{n-1}}
\end{aligned} \tag{38}$$

The previous system implies that

$$\begin{aligned}
\ddot{\theta}_1 &> \underline{\theta} \\
\ddot{\theta}_i - \ddot{\theta}_{i-1} &= \frac{p_i}{q_{i+1} - q_i} + \frac{p_{i-1}}{q_i - q_{i-1}} + \frac{p_i - p_{i-1}}{q_i - q_{i-1}} > \ddot{\theta}_{i-1} - > \ddot{\theta}_i > 2\ddot{\theta}_{i-1} \\
\bar{\theta} - \ddot{\theta}_{n-1} &= \frac{p_n - p_{n-1}}{q_n - q_{n-1}} + \frac{p_n}{q_n - q_{n-1}} > \ddot{\theta}_{n-1} - > \bar{\theta} > 2\ddot{\theta}_{n-1}
\end{aligned}$$

$\underline{\theta} > 0$ implies that $\ddot{\theta}_i > 0$. Then we can multiply the l.h.s. of the previous inequalities and the r.h.s and the sign of the inequality is preserved to yield:

$$\prod_{i=1}^{n} \ddot{\theta}_i > 2^{n-1} \prod_{i=0}^{n-1} \ddot{\theta}_i$$

After simplifying we get:

$$\bar{\theta} > 2^{n-1} \underline{\theta} \tag{39}$$

If (39) is not satisfied, there is no equilibrium such that the n firms produce a positive quantity. If we let $n^* = int[\bar{n}]$ such that $\frac{\bar{\theta}}{\underline{\theta}} = 2^{\bar{n}-1}$, we have that there can not be an equilibrium with more than n^*active firms. Therefore in an entry game, even if the fixed cost tended to zero, we would not have infinite firms. The number of firms that can survive in this market is bounded above by n^*. This situation is known as a natural oligopoly. We do not converge to the competitive situation. The number of firms that can survive in this market is determined by the heterogeneity of consumers. The greater the heterogeneity, the greater the number of firms that can survive in this market. (Observe that this result is independent of the qualities offered by firms).

In general one can prove (Shaked and Sutton (1983)) that natural oligopolies arise whenever consumers agree on the ranking of goods not at constant prices but at unit costs, that may be increasing in quality (in the previous case the condition held because unit costs did not depend on quality). It is easy to find the intuition of the result. Contrarily assume that when the fixed cost tends to 0 we have infinite firms. Then firms are selling basically undifferentiated good and applying the Bertrand logic, prices should tend to unit costs. But in this case only one firm makes sales and the others would prefer no to have entered the market.

We are going to see how this condition can be rewritten in terms of our model. Let $c(q)$ be the unit cost as a function of quality q. The utility of a consumer that buys a good of quality q at a price $c(q)$ is given by:

$$V(q) = -c(q) + \theta q$$

All consumers will have the same preferences about the goods of different quality sold at unit costs if the following derivative have the same sign for all consumers (i.e it does not depend on θ):

$$V'(q) = -c'(q) + \theta$$

This will be the case if either $c'(q) < \underline{\theta}$ (all prefer the highest quality) or $c'(q) > \bar{\theta}$ (all prefer the lowest quality).

The first condition has an interesting interpretation. Natural oligopolies will arise when unit costs are not very important and the differences between qualities are explained by different fixed costs, like, for example, R&D or advertising expenditures. Sutton (1992) supports empirically this results because he shows that markets where firms compete on advertising outlays have the nature of natural oligopolies.

6 Repeated games.

So far we have studied static oligopoly games. We are going to see what happens when those games are repeated several times. Assume that in the static game the strategy space of each firm i is A_i and its profits are given by $\pi_i(a_i, a_{-i})$. $\hat{a}$ is the unique Nash equilibrium of the game. This static game is repeated in T periods, where a_{it} denotes the action taken by firm i in period t. Firms discount the future at the discount factor δ. The objective function of firms is:

$$\sum_{t=0}^{T-1} \delta^t \pi_i(a_{it}, a_{-it})$$

There is no physical relation between the periods but we allow that the strategy of each player in each period be a function of the actions taken by all players in the previous periods.

We require that the strategies conform a subgame perfect equilibrium i.e. for any history up to period T, the strategy of each firm should maximize its profits from period T on given the strategies of its competitors from period T on.

We obtain the equilibrium of the game by applying backward induction. In the last period there is no future, and every firm only cares about the profits in this period. Then the objective function is like in the static case and we will have therefore the same equilibrium. In the next to last period, there is future of one period but it is fixed for what we have just seen. Therefore, firms only care about the profits in this period and they will play the same actions as in the static case. The same argument can be applied recursively for all periods up to the first. Therefore, the equilibrium of the repeated game will be simply the repetition T times of the equilibrium of the static game. Things change, however, dramatically when the game is repeated infinite times. In this case, there is always future and the previous backward induction argument can not be applied.

However, playing in every period the equilibrium of the repeated game is still an equilibrium. It is sustained by the following strategies played by all firms "play the action of the static equilibrium in every period independently of the actions of the competitors in the past". In other words, firms do not condition their actions to the actions taken by competitors in the past. In this case, as the future play of rivals is independent of how they play today, they only care about current profits and they optimally take the same actions as in the static case.

Do other equilibria exist? In particular, is it possible to support equilibrium payoffs that dominate the results obtained in the static game? In the equilibrium of the static game, industry profits are not maximized. Therefore, there is room for increasing payoffs if firms are able to sustain cooperation. This requires that deviation from cooperation has a cost in the form of lower continuation payoffs. In plain words, that a firm that fails to cooperate will be punished in the future. As the punishment is part of the equilibrium it should conform a subgame perfect equilibrium from the moment it starts to be applied. For the time being, we can use as punishment playing the static equilibrium in every period. It has the advantage that it is a play that we know that can be sustained in a subgame perfect equilibrium.

Therefore the strategy of players would be to play a cooperative strategy a_i^* (yielding a payoff of π_i^*) while the competitors have played the cooperative strategy in the past. Otherwise, firms play the action of the static equilibrium $\hat{a}_i$ (yielding a payoff of $\hat{\pi}_i$). This is usually called Nash reversion in case of deviation. This strategy can be formally written as:

$$\begin{aligned} & a_{i0} = a_i^* \\ & a_{it} = a_i^* \text{ if for any } \tau < t \text{ and } j = 1...n \; a_{j\tau} = a_j^* \\ & a_{it} = \hat{a}_i \text{ otherwise} \end{aligned} \tag{40}$$

To see if they conform an equilibrium, one should define $\overline{\pi}_i = Max_{a_i \in A_i}\ \pi_i(a_i, a^*_{-i})$. Then strategies (40) are an equilibrium of the repeated game if for all i we have that:

$$\overline{\pi}_i + \frac{\delta \hat{\pi_i}}{1-\delta} \leq \frac{\pi^*_i}{1-\delta} \tag{41}$$

In other words, the discounted profits of playing the cooperative action is greater than the profits obtained by deviating and having to suffer the punishment in all future periods. (41) can be rewritten as:

$$\overline{\pi}_i - \pi^*_i \leq \frac{\delta}{1-\delta}(\pi^*_i - \hat{\pi_i})$$

The gains today from deviation are lower than the discounted future losses.

$$\delta \geq \frac{\overline{\pi}_i - \pi^*_i}{\overline{\pi}_i - \hat{\pi_i}} \tag{42}$$

The r.h.s. of the previous inequality is lower than 1, because $\pi^*_i > \hat{\pi_i}$. The cooperative[7] equilibrium can be sustained if the future is important enough i.e. the future is not discounted too much.

Within the same formulation, δ can be interpreted as the probability in a period that the market survives one period more. This makes clear that what matters for the result is not that the game lasts forever but that one can never exclude that the market will still be going in the following period. In other words, cooperation is driven by the perpetual presence of future.

To illustrate the result we are going to focus on the case of Bertrand competition, homogeneous goods and constant marginal costs. The cooperative agreement consists on every firm choosing the monopoly price and sharing equally demand. In this way, each firm obtains the nth part of the monopoly profits (Π^M). The optimal deviation consists in undercutting the rivals and obtaining the full monopoly profits. Therefore, (42) can be rewritten as:

$$\delta \geq \frac{\Pi^M - \Pi^M/n}{\Pi^M} = 1 - \frac{1}{n} = \delta^B_n$$

In the problem set you are asked to do the same exercise for the case of Cournot competition. You would also find a cutoff δ^C_n. The comparison between both cutoffs is ambiguous a priori because in Bertrand the gains from deviation are greater but also the punishment is harder. The first change would increase the r.h.s.of (42) and the second change would decrease it. In fact we have in duopoly $\delta^B_n < \delta^C_n$ but $\delta^B_n > \delta^C_n$ otherwise. Both cutoffs δ^B_n and δ^C_n are increasing in n. This means that cooperation (collusion) becomes more difficult when the number of firms increases. This adds a new motivation to mergers: to facilitate collusion.

[7] Cooperation among firms in market games is usually known as collusion, because it negatively affect a third party i.e. consumers.

6.1 Variability of demand.

Assume we have n firms that compete in prices in a market with stochastic demand that can be high or low with the same probability. Firms observe the realization of demand before deciding the level of prices. We are going to study the maximal level of collusion that can be sustained. We focus on stationary equilibria: all firms charge the price p_h in all high demand periods and p_l in the low demand periods. When a firm deviates from those prices, the equilibrium stipulates that all firms price at marginal cost forever. Denote respectively π_h and π_l as the industry profits when firms charge p_h and p_l respectively. Then, the incentive compatibility constraint with high demand is:

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{\pi_h+\pi_l}{2n}\right) \geq \frac{n-1}{n}\pi_h \tag{43}$$

and with low demand:

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{\pi_h+\pi_l}{2n}\right) \geq \frac{n-1}{n}\pi_l \tag{44}$$

One obtains collusion is more difficult in high-state demands. The losses from deviation are the same in both states, but the gains increase with demand size.

One can check that if $\delta < \dfrac{n-1}{n}$, the only solution to (65) and (66) supposes zero profits for both firms in both states of demand. Adding (65) and (66) we have:

$$\left(\frac{\pi_h+\pi_l}{n}\right)\left(\frac{\delta}{1-\delta}-(n-1)\right) \geq 0$$

$$\left(\frac{\pi_h+\pi_l}{n}\right)\left(\frac{\delta-(n-1)(1-\delta)}{1-\delta}\right) \geq 0$$

$$\left(\frac{\pi_h+\pi_l}{n}\right)\left(\frac{\delta-(n-1)(1-\delta)}{1-\delta}\right) \geq 0$$

$$\left(\frac{\pi_h+\pi_l}{n}\right)\left(\frac{n\delta-(n-1)}{1-\delta}\right) \geq 0$$

$$(\pi_h+\pi_l)\left(\frac{\delta-\frac{n-1}{n}}{1-\delta}\right) \geq 0$$

Then for $\delta < \dfrac{n-1}{n}$, the unique solution is $\pi_h = \pi_l = 0$. If $\delta \geq \dfrac{n-1}{n}$, whenever $\pi_h \geq \pi_l$ (66) holds and increases in π_l ease constraint (65). Then the maximal level of collusion will be obtained by choosing $\pi_h = \bar{\pi}_h \geq \pi_l = \pi_l^m$, where π_l^m are the monopoly profits with low demand and $\bar{\pi}_h$is the maximal level of profits in high demand compatible with (65). Define π_h^* as the number that satisfies (65) with equality :

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{\pi_h^*+\pi_l^m}{2}\right) = (n-1)\pi_h^*$$

$$\frac{\delta\pi_l^m}{2(n-1)-(2n-1)\delta} = \pi_h^*$$

As profits can not be higher than monopoly profits π_h^m, we have that

$$\begin{aligned}
\bar{\pi}_h &= \pi_h^* \text{ if } \frac{n-1}{n} \leq \delta \leq \frac{2(n-1)\pi_h^m}{(2n-1)\pi_h^m + \pi_l^m} \\
\bar{\pi}_h &= \pi_h^m \text{ if } \underline{\delta} = \frac{2(n-1)\pi_h^m}{(2n-1)\pi_h^m + \pi_l^m} < \delta
\end{aligned}$$

(Observe that

$$\frac{\delta\pi_l^m}{2(n-1)-(2n-1)\delta} > \pi_h^m if \frac{2(n-1)\pi_h^m}{(2n-1)\pi_h^m + \pi_l^m} < \delta.$$

) Only in this last case one can sustain monopoly profits in all states of demand. We have sign$[\frac{\partial\underline{\delta}}{\partial n}] = 2((2n-1)\pi_h^m + \pi_l^m) - 4(n-1)\pi_h^m = 2(\pi_h^m + \pi_l^m) > 0$. In other words, reductions in the number of firms facilitate collusion.

We have seen that collusion is more difficult to sustain when demand is high. This may lead to the case where prices are lower when demand is high. If this is the case, we would observe price wars in booms. In two sections, we are going to deal with a model with opposite predictions.

6.2 Multimarket contact

We have studied vertical and horizontal mergers and we have seen their main motivations. Another frequent type of mergers are conglomerate mergers, where firms expand to areas unrelated to their original activities. Those mergers are more difficult to explain but the following model brings an explanation. Conglomerate mergers would occur because firms want to face their competitors in more than one market. We are going to see that the fact that a firm competes with other firms in more than one market (even if unrelated) broadens the scope for collusion.

Different detection lags. We are going to illustrate the effect of multimarket contact with the following example. Assume that we have two markets. They have the same demand and costs. The only difference is that in one market the prices of one period are known the following period and in the other market two periods after. In other words, deviations will be detected in one market the following period and in the market in two periods. If firms in each market are different, in the first market we saw that collusion can be sustained if

$$\begin{aligned}
\pi^m &\leq \frac{\pi^m}{n(1-\delta)} \qquad (45)\\
\delta &\geq \frac{n-1}{n}
\end{aligned}$$

and in the second, collusion can be sustained if:

$$\begin{aligned} (1+\delta)\pi^m &\leq \frac{\pi^m}{n(1-\delta)} \\ \delta &\geq \sqrt{\frac{n-1}{n}} \end{aligned} \tag{46}$$

What would it happen if the same n firms operated in the two markets? In this case, the collusive aggreement would take both markets into consideration. Any deviation in any market would trigger Nash reversion in both markets. Which is the new condition that guarantees that full collusion can be sustained in both markets? [As if collusion can not be sustained in the first market there is no hope that it can be sustained jointly in both markets, one should only worry about the case $\delta \geq \dfrac{n-1}{n}$] The optimal deviation consists in deviating first in the second market and in the following peridod deviating in both markets[8].

The incentive compatibility constraint will be:

$$(1+2\delta)\pi^m + \frac{\pi^m}{n} \leq \frac{2\pi^m}{n(1-\delta)}$$

The l.h.s. represent the deviation profits. In the first period, the deviator deviates in the first market and obtains π^m and keeps operating in the second market getting $\dfrac{\pi^m}{n}$. In the second period, he deviates in both markets getting $2\delta\pi^m$. In the third period, it obtains nothing because firms revert to the Bertrand equilibrium. The r.h.s. represents the profits of cooperation as usual.

This can be rewritten as:

$$(1+2\delta)\pi^m \leq \frac{\pi^m(1+\delta)}{n(1-\delta)} \tag{47}$$

The key point is that the new constraint is the sum of constraints (45)-discounted one period- and (46). If (45) and (46) hold, (47) holds for sure. But it may be the case-this is the crucial point-that (47) holds and (46) does not hold. The intuition is that the loss of collusion in the first market may be so large as to deter deviations not only in this market but also in the second market. (Technically the incentive (no-undercutting) constraints are pooled into a single constraint). (47) holds if:

$$\delta \geq \bar{\delta} = \frac{\sqrt{1-10n+9n^2}+n-1}{4n}$$

We have that $\sqrt{\dfrac{n-1}{n}} > \bar{\delta}$ and then multimarket contact facilitates collusion. Observe that we have that $\dfrac{\partial\bar{\delta}}{\partial n} > 0$.

[8] Given that we are only considering the case of high discount factor, this yields more profits than deviating already in both markets in the first period $(1+2\delta)\pi^m \geq (1+\frac{2(n-1)}{n})\pi^m > 2\pi^m$.

Variable demand Assume that we have n firms operating in 2 markets. Demand in both markets can be high or low with the same probability. Given the state, the demand is the same in both markets, but we assume that demands are perfectly negatively correlated i.e.if demand is low in one market is high in the other market. Collusion can be sustained in both markets if:

$$
\begin{aligned}
(\frac{\delta}{1-\delta})\frac{\pi_l^m+\pi_h^m}{n} &\geq (\pi_l^m+\pi_h^m)\frac{n-1}{n} \qquad (48)\\
\delta &\geq \frac{n-1}{n}
\end{aligned}
$$

Therefore multimarket contact facilitates collusion. Observe that (48) is the sum of the incentive constraint when demand is high (65) and when it is low (66). (48) may hold even if either (65) or (66) does not hold.

6.3 Imperfect observability.

So far we have studied models where the actions of competitors were perfectly observable. Now we are going to see a model where the actions of competitors are not directly observable, but a public signal influenced by the actions is available. The original papers developing this idea are Green and Porter (1984) and Porter (1983), where firms are assumed to choose quantities. However in class we are going to consider the simpler model in Tirole (1988) where firms choose prices in a infinitely repeated game.

We have the following stochastic demand in each period:

$$P_t = p(Q_t)\theta_t$$

θ_t takes the value 0 with probability α and the value 1 with probability $1-\alpha$. We are going to assume that firms can not observe the prices set by competitors. They can only observe the sales they individually make. Therefore, they do not know if they make no sales because demand is low or because some rival has cheated on the agreement.

We are going to see when the following strategies conform an equilibrium: firms set the monopoly price whenever they have sold something in the previous period. When they get no demand, the price at marginal cost for T periods.

We are going to calculate the present discounted value V of following this strategy when they are setting the monopoly price. With probability $(1-\alpha)$ demand will be high and they will obtain today the nth part of the monopoly profits. In the next period, they will still set the monopoly price and obtain a discounted profit of V. With probability α demand will be low and they will play the static equilibrium for T periods. After those periods, they will again set the monopoly price with a discounted present value of V. Those explanations are summarized in the following expression:

$$
\begin{aligned}
V &= (1-\alpha)(\Pi^m/n+\delta V)+\alpha\delta^{T+1}V \qquad (49)\\
V &= \frac{(1-\alpha)\Pi^m/n}{1-(1-\alpha)\delta-\alpha\delta^{T+1}}
\end{aligned}
$$

We are going to check under which conditions those strategies conform an equilibrium. This will be the case if no firm wants to deviate either in the punishment or collusive phase. No deviation is profitable in the punishment phase because firms are maximizing per-period profits and the behavior of competitors does not depend on how they behave during the punishment phase. So we only have to find conditions such that firms do not want to deviate during the collusive phase. Deviation in the collusive phase is not profitable if the following condition holds:

$$\begin{aligned} V &\geq (1-\alpha)(\Pi^m + \delta^{T+1}V) + \alpha\delta^{T+1}V \quad (50) \\ V(1-(1-\alpha)\delta^{T+1} - \alpha\delta^{T+1}) - (1-\alpha)\Pi^m &\geq 0 \\ V(1-\delta^{T+1}) - (1-\alpha)\Pi^m &\geq 0 \\ 1-\delta^{T+1} - n(1-(1-\alpha)\delta - \alpha\delta^{T+1}) &\geq 0 \\ \phi(T) &= 1-n+n(1-\alpha)\delta + (n\alpha-1)\delta^{T+1} \geq 0 \end{aligned}$$

$$\phi(0) = -(n-1)(1-\delta) < 0$$

Obviously, without punishment, collusion can not be sustained, because there are always incentives to cheat. We have that

$$\begin{aligned} \phi(T+1) - \phi(T) &> 0 \\ \phi(T+1) - \phi(T) &= \delta^T(1-n\alpha)(1-\delta) \end{aligned}$$

Then if $1 \leq n\alpha$ $\phi(T)$ is decreasing and (50) can not be satisfied. The intuition for this is that the temptation to undercut increases when the expected gains from future collusion decrease.

Then a necessary condition for (50) to be satisfied is:

$$1 > n\alpha \quad (51)$$

A sufficient condition for (50) to be satisfied is that:

$$\underset{T->\infty}{Lim}\phi(T) = 1-n+n(1-\alpha)\delta > 0$$

In other words, it must be the case that

$$(1-\alpha)\delta > \frac{n-1}{n} \quad (52)$$

(52) generalizes the result with deterministic demand which corresponds to $\alpha = 0$ and $T = \infty$ (no price war with no deviation and the duration of the price war in case of deviation is infinite).

Asuming that (52) holds, there exist equilibria of the type we are considering. In this case which T will be chosen by firms? The one that maximizes their payoffs (49). As it is decreasing in T, they will choose the minimal T that satisfies (50). In the case of deterministic demand, the length of the punishment did not matter because it never occurred in equilibrium. Now it is important, punishment do occur in equilibrium and to maximize their value firms wants to reduce its length. In playing this equilibrium, we would observe that firms start a price war (price at marginal cost) after a downturn of demand.

6.3.1 Green and Porter (1984)

In particular, we are going to analyze Green and Porter (1984) and Porter (1983) where firms choose quantities and observe the price after having chosen the quantities. Price is an imperfect signal of the quantities because demand is stochastic. Firms will condition their action in one period to the only available information that is the sequence of past prices.

Formally the assumptions of the model are the following. Inverse demand in period t (demand changes in every period because it is stochastic) is given by:

$$P_t = p(Q_t)\theta_t$$

where θ_t is an independent shock, identically distributed across periods with distribution function $F(.)$. Firms are symmetric. $\pi(q)$ denotes the expected single-period profits when every firm produces q.

We check when a collusive equilibrium where each firm produces a quantity q^* (lower than the Cournot output q^C) can be sustained using the following strategies. Every firm produces q^* until the price is lower than a certain cutoff $\bar{p}$. Then all firms produce q^C for T periods, returning to the collusive output after that, and all the process re-starts again.

We are going to calculate the discounted present value of playing these strategies (if they are also followed by competitors) if we are in a period where firms play the collusive output. In the first period, firms obtain $\pi(q^*)$. Price will be lower than the cutoff if $\bar{p} > \theta_t p(nq^*)$ i.e. $\theta_t < \dfrac{\bar{p}}{p(nq^*)}$. This happens with probability $\alpha = F(\dfrac{\bar{p}}{p(nq^*)})$. In the second period, with probability $1-\alpha$ they will produce the collusive output with a discounted profits of V. With probability α, we have a price war and players produce q^C for T periods. After T periods, they again play the collusive output and discounted profits are again given by V.

$$\begin{aligned}
V &= \pi(q^*) + \delta(1-\alpha)V + \delta\alpha\left(\delta^T V + \sum_{s=0}^{T-1}\delta^s\pi(q^C)\right) \\
V &= \pi(q^*) + \delta(1-\alpha)V + \delta\alpha\left(\delta^T V + \frac{1-\delta^T}{1-\delta}\pi(q^C)\right) \\
\text{where } \alpha &= F(\frac{\bar{p}}{p(nq^*)}) \text{ is the probability of a price war}
\end{aligned}$$

Then we have:

$$\begin{aligned}
V(1-\alpha\delta^{T+1}-(1-\alpha)\delta) &= \pi(q^*) - \pi(q^C) + \pi(q^C)\left(1 + \frac{\alpha\delta(1-\delta^T)}{1-\delta}\right) \\
V(1-\alpha\delta^{T+1}-(1-\alpha)\delta) &= \pi(q^*) - \pi(q^C) + \pi(q^C)\frac{1-\delta+\alpha\delta(1-\delta^T)}{1-\delta}
\end{aligned}$$

$$V = \frac{\pi(q^*) - \pi(q^C)}{1 - \alpha\delta^{T+1} - (1-\alpha)\delta} + \frac{\pi(q^C)}{1-\delta} \tag{53}$$

The present value of playing the equilibrium strategies exceeds the discounted Cournot profits by the amount $\pi(q^*) - \pi(q^C)$ appropriately discounted both by the discount factor δ and by the occasional interferences of the punishment phases of duration T.

We are going to check under which conditions those strategies conform an equilibrium. This will be the case if no firm wants to deviate either in the punishment or collusive phase. No deviation is profitable in the punishment phase because firms are maximizing per-period profits and the behavior of competitors does not depend on how they behave during the punishment phase. So we only have to find conditions such that firms do not want to deviate during the collusive phase. For that we compute the discounted value of playing q_i when the candidate equilibrium prescribed to play q^*. It amounts to:

$$\begin{aligned} V(q_i, q^*) &= \pi_i(q_i, q^*) + \delta(1-\alpha(q_i))V + \delta\alpha(q_i)\left(\delta^T V + \frac{1-\delta^T}{1-\delta}\pi_i(q^C)\right) \\ \text{where } \alpha(q_i) &= F(\frac{\bar{p}}{p(q_{i+}(n-1)q^*)}) \end{aligned}$$

$\pi_i(q_i, q)$ denote the expected profits when firm i plays q_i and every other firm produce q.

The no deviation condition is given by:

$$\frac{\partial V}{\partial q_i}(q^*, q^*) = 0 \tag{54}$$

It can be written as:

$$\begin{aligned} \frac{\partial \pi(q^*, q^*)}{\partial q_i} - \delta\frac{\partial \alpha}{\partial q_i}(V - \delta^T V - \frac{1-\delta^T}{1-\delta}\pi_i(q^C)) &= 0 \\ \frac{\partial \pi(q^*, q^*)}{\partial q_i} - \delta\frac{\partial \alpha}{\partial q_i}(1-\delta^T)(V - \frac{\pi_i(q^C)}{1-\delta}) &= 0 \end{aligned}$$

$$\delta\frac{\partial \alpha}{\partial q_i}\frac{\pi(q^*) - \pi(q^C)}{1 - \alpha\delta^{T+1} - (1-\alpha)\delta}(1-\delta^T) = \frac{\partial \pi(q_i^*, q_{-i}^*)}{\partial q_i}$$

In other words, the gains today of deviating should be equal to the losses due to the fact that the probability of a price war increases.

The optimal cartel agreement is a triple $(q^*, \bar{p}, T)$ that maximizes V subject to (54).

$$\begin{aligned} & \underset{\{q^*,\bar{p},T\}}{Max} V \\ s.a.\ 0 &= \frac{\partial V}{\partial q_i}(q^*, q^*) \end{aligned}$$

Porter (1983) solves explicitly this model for the case of linear demand and constant marginal costs for different distribution functions of the shock. Then, for example, when

$$\begin{aligned} F(\theta) &= \left(\frac{\alpha\theta}{\alpha+1}\right)^{\alpha} \text{ for } 0 \leq \theta \leq \frac{\alpha+1}{\alpha},\ \alpha > 0 \\ E[\theta] &= 1 \text{ and } \mathrm{Var}[\theta] = \frac{1}{\alpha(\alpha+2)}. \end{aligned}$$

we have that

$$\begin{aligned} q^* &= q^m \left(\frac{N+\alpha+\frac{(N+1)a}{a-c}}{N+1+\alpha}\right) \text{ if } \alpha \geq \alpha^0 \\ &= q^C \text{ if } \alpha < \alpha^0 \end{aligned}$$

This result gives us the main message of the paper. When α increases, the precision of the signal increases and decreases the collusive quantity becoming closer to the monopoly output. When α tends to infinity, the distribution of θ becomes degenerate and $q^m = q^*$. When α is very low, the signal has a lot of noise and no collusive output can be sustained.

6.4 Optimal punishments

The severity of the punishment in case of deviation limits the extent of collusion. The harsher the punishement, the greater the incentives to collude. So far we have studied punishments consisting in reverting to the static Nash equilibrium. Now we are going to consider optimal punishments in the sense that they are the hardest punishments that can be sustained as a subgame perfect equilibrium. Following Abreu (1986, 1988), in a symmetric context, they prove that the equilibrium with optimal punishments is very simple, because it consists of a collusive output (the same for all firms) x_1 and a punishment output x_2. One produces x_1, whenever there is no deviation. If some deviates, all firms produce x_2 in one period, to produce again x_1 in the following period. When they play x_1, they are said to be in the carrot stage whereas when they play x_2 are said to be in the stick stage. Those strategies can be written formally as:

$$\begin{array}{l} a_{i0} = x_1 \\ a_{it} = x_1 \text{ if for any } j\ a_{jt-1} = x_1 \\ a_{it} = x_1 \text{ if for any } j\ a_{jt-1} = x_2 \\ a_{it} = x_2 \text{ otherwise} \end{array} \tag{55}$$

Defining $\pi(x)$ as the profits when all firms play x, the discounted value of the punishement is:

$$W = \pi(x_2) + \frac{\delta}{1-\delta}\pi(x_1) \tag{56}$$

For (55) to be a subgame perfect equilibrium, firms must not be willing to deviate either in the carrot stage or the stick stage. If $\bar{\pi}(x)$ denotes the one-period deviation profits when the other firms produce x, in the first case, it means that

$$\begin{aligned} W &\geq \bar{\pi}(x_2) + \delta W \\ (1-\delta)W &\geq \bar{\pi}(x_2) \end{aligned} \tag{57}$$

and in the second case:

$$\pi(x_1)\frac{\delta}{1-\delta} \geq \bar{\pi}(x_1) + \delta W \tag{58}$$

Assume that the harshest punishment that can be imposed to players is that they obtain zero profits

$$W = 0 \tag{59}$$

. They could always avoid negative profits by simply not producing[9]. Then (57) can be rewritten as:

$$0 \geq \bar{\pi}(x_2) \tag{60}$$

and (58) as

$$\pi(x_1)\frac{\delta}{1-\delta} \geq \bar{\pi}(x_1) \tag{61}$$

We are going calculate the equilibrium for the case where n firms compete a la Cournot, demand is given by $P = a - Q$ and the marginal cost is c. We have that $\pi(x) = (a - c - nx)x$. We are going to see for which values of the discount factor the monopoly outcome can be sustained in equilibrium. Denote by x^m the nth part of the monopoly output. Then in this case ($x_1 = x^m$).

Restrictions (59), (60) and (61) become then:

$$\pi(x_2) + \frac{\delta}{1-\delta}\pi(x^m) = 0 \tag{62}$$

$$x_2 \geq \frac{a-c}{n-1} \tag{63}$$

$$\pi(x^m)\frac{\delta}{1-\delta} \geq \left(\frac{(a-c)(n+1)}{4n}\right)^2 \tag{64}$$

(63) means that with the output of competitors price is already lower han marginal costs. Then, the optimal deviation output is zero and $\bar{\pi}(x_2) = 0$. (62) and (63) are satisfeid if $\delta \geq \dfrac{4n}{(n+1)^2} = \underline{\delta}$. The restriction is obtained from equation:

$$\pi\left(\frac{a-c}{n-1}\right) + \frac{\underline{\delta}}{1-\underline{\delta}}\pi(x^m) = 0$$

[9] In the Bertrand case, reversion to the Nash equilibrium gives us already an optimal punishment.

For higher x_2, a greater δ would be needed for (62) to be satisfied. (64) is satisfied if $\delta \geq \left(\frac{n-1}{n+1}\right)^2 = \bar{\delta}$. Then, the agreement can be sustained if $\delta \geq \max\{\underline{\delta}, \bar{\delta}\}$.

$\max\{\underline{\delta}, \bar{\delta}\}$ is nonmonotonic with respect to the number of firms. It is decreasing for $n \leq 5$ and increasing otherwise. The anomalous behavior appears when the binding constraint is the one of the stick stage (this constraint did not exist in the case of Nash reversion). In this case, it is easier to implement the punishment the higher the number of possible "punishers". This implies that the payoff in the stick stage is increasing in n ($n\frac{a-c}{n-1}$ is decreasing in n).

6.5 Variability of demand.

Assume we have n firms that compete in prices in a market with stochastic demand that can be high or low with the same probability. Firms observe the realization of demand before deciding the level of prices. We are going to study the maximal level of collusion that can be sustained. We focus on stationary equilibria: all firms charge the price p_h in all high demand periods and p_l in the low demand periods. When a firm deviates from those prices, the equilibrium stipulates that all firms price at marginal cost forever. Denote respectively π_h and π_l as the industry profits when firms charge p_h and p_l respectively. Then, the incentive compatibility constraint with high demand is:

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{\pi_h+\pi_l}{2n}\right) \geq \frac{n-1}{n}\pi_h \tag{65}$$

and with low demand:

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{\pi_h+\pi_l}{2n}\right) \geq \frac{n-1}{n}\pi_l \tag{66}$$

Intuitively one can check that the incentives to deviate are higher when demand is high. The losses are and average of the profits with low and high demand and therefore they are smaller than if the high demand persisted with certainty in the future.

One can check that if $\delta < \frac{n-1}{n}$, the only solution to (65) and (66) supposes zero profits for both firms in both states of demand. Adding (65) and (66) we have:

$$\left(\frac{\pi_h+\pi_l}{n}\right)\left(\frac{\delta}{1-\delta}-(n-1)\right) \geq 0$$

$$\left(\frac{\pi_h+\pi_l}{n}\right)\left(\frac{\delta-(n-1)(1-\delta)}{1-\delta}\right) \geq 0$$

$$\left(\frac{\pi_h+\pi_l}{n}\right)\left(\frac{\delta-(n-1)(1-\delta)}{1-\delta}\right) \geq 0$$

$$\left(\frac{\pi_h+\pi_l}{n}\right)\left(\frac{n\delta-(n-1)}{1-\delta}\right)\geq 0$$

$$(\pi_h+\pi_l)\left(\frac{\delta-\frac{n-1}{n}}{1-\delta}\right)\geq 0$$

Then for $\delta<\dfrac{n-1}{n}$, the unique solution is $\pi_h=\pi_l=0$. If $\delta\geq\dfrac{n-1}{n}$, whenever $\pi_h\geq\pi_l$ (66) holds and increases in π_l ease constraint (65). Then the maximal level of collusion will be obtained by choosing $\pi_h=\bar{\pi}_h\geq\pi_l=\pi_l^m$, where π_l^m are the monopoly profits with low demand and $\bar{\pi}_h$is the maximal level of profits in high demand compatible with (65). Define π_h^* as the number that satisfies (65) with equality :

$$\left(\frac{\delta}{1-\delta}\right)\left(\frac{\pi_h^*+\pi_l^m}{2}\right)=(n-1)\pi_h^*$$

$$\frac{\delta\pi_l^m}{2(n-1)-(2n-1)\delta}=\pi_h^*$$

As profits can not be higher than monopoly profits π_h^m, we have that

$$\begin{aligned}\bar{\pi}_h &= \pi_h^* \text{ if } \frac{n-1}{n}\leq\delta\leq\frac{2(n-1)\pi_h^m}{(2n-1)\pi_h^m+\pi_l^m}\\ \bar{\pi}_h &= \pi_h^m \text{ if } \underline{\delta}=\frac{2(n-1)\pi_h^m}{(2n-1)\pi_h^m+\pi_l^m}<\delta\end{aligned}$$

(Observe that

$$\frac{\delta\pi_l^m}{2(n-1)-(2n-1)\delta}>\pi_h^m if \frac{2(n-1)\pi_h^m}{(2n-1)\pi_h^m+\pi_l^m}<\delta.$$

) Only in this last case one can sustain monopoly profits in all states of demand. We have sign$[\frac{\partial\underline{\delta}}{\partial n}]=2((2n-1)\pi_h^m+\pi_l^m)-4(n-1)\pi_h^m=2(\pi_h^m+\pi_l^m)>0$. In other words, reductions in the number of firms facilitate collusion.

We have seen that collusion is more difficult to sustain when demand is high. This may lead to the case where prices are lower when demand is high. We illustrate the case with an example with linear demand $X=a_j-P$ where $j=h,l$ and $a_h>a_l$. For values such that full collusion can not be sustained, the price with high demand satisfies:

$$\frac{\delta(\frac{a_l}{2})^2}{2(n-1)-(2n-1)\delta}=(a_h-p_h)p_h$$

It is an involved expression but one can check that it is lower than $p_l^m=\dfrac{a_l}{2}$ whenever $\dfrac{2a_h-a_l}{(1+\frac{n}{n-1})a_h-a_l}>\delta\geq\dfrac{n-1}{n}$. Therefore, we would observe prices wars in booms. In two sections, we are going to deal with a model with opposite predictions.