Solving Dynamic Games by Discretizing the State Distribution

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April 2009

Abstract

After more then a decade of advances in theoretical models for industry dynamics, there has been a growth in applications using recently developed estimation methods for dynamic games. However, estimation methods often require equilibrium calculation either in the estimation routine and/or for the construction of counterfactuals, and it is well known that, due to the 'curse of dimensionality ', equilibrium calculations can be computationally demanding .

In this paper a method to circumvent the 'curse of dimensionality' is proposed. Using the result that under symmetry and anonymity the industry distribution fully characterizes the industry state, we can use quantiles to discretize and approximate this distribution. As the number of quantiles increases, the approximation becomes exact.

This resembles the widely used discretization method in solving dynamic programming problems with continuous state variables. In the dynamic game case, the continuous state variable is a distribution, defined over the unit interval. The main advantage of this approach is that the dimension of the state space becomes unrelated with the number of players. Using simulations it is illustrated that the methodology sucessfully reduces the cardinality of the problem while providing a good approximation.

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1 Introduction

There has been a recent surge in the estimation of dynamic games[1] after the development of estimation methods (Aguirregabiria and Mira, 2007; Bajari Benkard and Levin, 2007; Pakes, Ostrovsky and Berry, 2007; Pesendorfer and Schmidt-Dengler, 2008). However, the application of these methods is still severely constrained by the 'curse of dimensionality' because of the exponential growth of the state space as either (i) the number of players or (ii) the number of state variables increase. Even though some methods are not directly affected by this problem by avoiding equilibrium calculations (Bajari, Benkard and Levin, 2007), they are not free from it when it comes to the construction of counterfactuals.

Several methods have been proposed. Pakes and Mcguire (2001) suggest a genetic algorithm which wanders through the state space and updates states in the recurrent class. The advantage is that the problem can be more easily solved if the recurrent class of the state space is significantly smaller than the full state space which makes this algorithm mostly fit for cases where the ergodic set has low cardinality.

Doraszelski and Judd (2008) propose a continuous time version of the discrete time game. In continuous time the possibility of simultaneous changes disappears and this allows the number of states over which integration is performed to grow linearly rather than exponentially. This type of framework is ideal for situations where simultaneous change is not possible or unlikely (e.g. sequential games).

Finally Weintraub, Benkard and Van Roy (2008) propose an alternative equilibrium concept, the 'oblivious equilibrium'. In the oblivious equilibrium players' strategies depend solely on individual state and long run state distribution. They show that this framework is a good approximation for industries with a large number of firms and no aggregate shocks, provided the industry distribution satisfies a light tail condition (i.e. no market leaders).

In this paper I propose an alternative approximation method for the cases where anonymity and symmetry are satisfied. Since in these cases the industry distribution fully characterizes the industry state, what I propose is the discretiza-

[1] A non-exhaustive list includes Aguirregabiria and Ho, 2009; Collard-Wexler, 2006; Hashmin and Van Biesebroeck, 2008; Ryan, 2006; Santos, 2008; Santos and Van Reenen, 2008; Schmidt-Dengler, 2006; Varela-Irimia, 2008; Xu, 2008.

tion of this industry distribution using quantiles. In this case an equilibrium is defined as a transition matrix for the quantiles of the industry distribution (for example $\Pr(P^1(s_{t+1}), P^2(s_{t+1}), ...P^R(s_{t+1})|P^1(s_t), P^2(s_t), ...P^R(s_t))$ is the quantile transition where s_t is the industry state and P^r is the rth quantile of the distribution of s). The method of discretization is common when using numerical solutions for dynamic programming problems with continuous state variables. In the dynamic game case, the continuous state variable is the distribution of competitors' states, defined over the unit interval.

Two main advantages arise. First, the dimensionality of the problem becomes completely unrelated with the number of firms which makes this particularly attractive for industries with more than a handful of firms. Second, the quality of the approximation does not depend on the number of firms in the market but on how "fine" the chosen quantile grid is (i.e. the number of quantiles used). This makes the method fit for the cases where researchers are interested in particular market structures since the quantiles can be chosen to fit the needs. I present evidence that using as little as five quantiles provides a reasonably good approximation while keeping the problem computationally tractable. Intuitively, for cases with large number of players there is less to be lost with such an approximation (see for example Weintraub, Benkard and Van Roy, 2008). However, the computational results show that even for problems with 10 firms the loss is relatively small. This illustrates one of the results that the goodness of the approximation does not depend on the number of players in the industry.

Comparing to the existing methods, using quantiles does not add the restrictions on the timing of actions as Doraszelski and Judd (2008). When compared with Weintraub et al (2008), the quantiles approach has an advantage since it has no restrictions on aggregate shocks and leading firms. Furthermore, the quality of the approximation is a variable that the researcher can choose by setting the number of quantiles and it is not dependent on the number of firms in the industry being very large. On the other hand the disadvantages are that first, it restricts to symmetric and anonymous equilibria and second, in practice for computational reasons the number of quantiles has to be kept small (normally less than 10).

In the next section I present the general model which follows closely Ericson and Pakes (1995). In section 3 I introduce the approximation and explain the main results. Section 4 contains simulations illustrating the goodness of the approximation and finally section 5 concludes.

2 Model

This section describes the elements of the general model. In particular I describe the sequencing of events, the period game, the transition function, the pay-offs, the strategies, and the equilibrium concept. I consider a dynamic game with discrete time $t = 1, 2, ..., \infty$. The number of players is N and is assumed fixed over time and a typical player is denoted by $i \in N$. Players can choose actions $a_{it} \in \mathfrak{A}$. I will focus on stationary Markov games and I will use the following notation for time, $\mathbf{s}_{t+1} = \mathbf{s}'$.

2.1 States, actions and state transition

States Each player is endowed with a state variable, $s_{it} \in \mathfrak{s} =\{1, 2, ..., K\}$. The industry state can be written as $\mathbf{s}_t = (s_{1t}, ..., s_{Nt}) \in \mathfrak{s}^N$.

Actions Each player chooses actions from a continuous set $a_{it} \in \mathfrak{A}_i$ and $\mathbf{a}_t = (a_{1t}, ...a_{Nt}) \in \mathfrak{A} = \times_{i=1}^{N} \mathfrak{A}_i$. All players choose actions simultaneously and are taken after observing the state $\mathbf{s}_t$.

State transition The state transition is described by a probability density function $q : \mathfrak{s}^N \times \mathfrak{s}^N \times \mathfrak{A}^N \rightarrow [0, 1]$ where a typical element $q(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$ equals the probability that state $\mathbf{s}_{t+1}$ is reached from state $\mathbf{s}_t$ when players choose actions $\mathbf{a}_t$. Also it is required that $\sum_{\mathbf{s}_{t+1} \in \mathfrak{s}^N} q(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t) = 1$. For simplicity, I will restrict to the case where there are no spillovers

Assumption 2.1 *(No Spillovers) Conditional on current state and actions, own state evolves according to the transition function*

$$p(s_i'|s_i, a_i)$$

2.2 Pay-offs

Period pay-off Player i's pay-off depend on all players actions and strategies $\pi(\mathbf{a}_t, \mathbf{s}_t)$ satisfying the usual regularity conditions (continuity and boundedness). Again for simplicity reasons I will restrict to the no spillovers case and $\pi(\mathbf{a}_t, \mathbf{s}_t) = \pi(a_{it}, \mathbf{s}_t)$

Game pay-off Players discount the future at rate $\rho \in [0, 1)$ and the game pay-off of player i is equal to the present discounted value of all future pay-offs.

Timing In each period t, the game has the following order:

1. States ($\mathbf{s}$) are commonly observed;
2. Actions ($\mathbf{a} = (a_1, .., a_N)$) are decided simultaneously;
3. Firms compete in the market and receive period returns, $\pi(a_{it}, \mathbf{s}_t)$;
4. Both stochastic and deterministic outcomes of actions are realized and new industry state is formed ($\mathbf{s}'$).

Value Function Firm i's individual problem can be written as

$$V_i(\mathbf{s}) = \max_{a_i \in \mathfrak{A}} \pi(a_{it}, \mathbf{s}) + \rho \sum_{\mathbf{s}' \in \mathfrak{s}^N} V(\mathbf{s}') q(\mathbf{s}' | \mathbf{s}, a_{i,} \mathbf{a}_{-i}), \ \forall i, \mathbf{s}$$

where $q(\mathbf{s}' | \mathbf{s}, a_{i,} \mathbf{a}_{-i})$ is the evolution of the industry state conditional on player i choosing action a_i and the other players choosing actions $\mathbf{a}_{-i} = (a_1, ...a_{i-1}, a_{i+1}, ..., a_N)$.

2.3 Strategies and equilibrium

Strategies For each state firms can take actions $a_i \in \mathfrak{A}_i$. I restrict to pure Markovian strategies. These strategies are a mapping from the set of pay-off relevant states onto the action space $a_i(\mathbf{s}) : \mathfrak{s}^N \to \mathfrak{A}_i$ where the action space is the compact set, $\mathfrak{A}_i$.

Equilibrium The equilibrium concept is Markov Perfect Equilibrium in the sense of Maskin and Tirole (1988, 2001). Since the focus of this paper is on approximation methods, I will abstract from problems related with existence as these have been studied in the literature (for example Doraszelski and Satterthwaite (2007) or Schmidt-Dengler and Pesendorfer (2008) provide existence proofs for a similar class of models).

Definition 2.2 (Equilibrium) *A collection of strategies* $(a_i^*(\mathbf{s}))$ *is an equilibrium if for any* i *and* $\mathbf{s}$

$$a_i^*(\mathbf{s}) = \arg\max_{a_i \in \mathfrak{A}} \pi(a_i, \mathbf{s}) + \rho EV_i(\mathbf{s}'), \ \forall i, \mathbf{s} \tag{1}$$

where $EV_i(\mathbf{s}') = \sum_{\mathbf{s}' \in \mathfrak{s}^N} V_i(\mathbf{s}') q(\mathbf{s}'|\mathbf{s}, a_i, \mathbf{a}_{-i})$

The optimal value function at time t can be written as

$$V_i(\mathbf{s}) = \max_{a_i \in \mathfrak{A}} \pi(a_{it}, \mathbf{s}) + \rho \sum_{\mathbf{s}' \in \mathfrak{s}^N} V_i(\mathbf{s}') q(\mathbf{s}'|\mathbf{s}, a_{i,} \mathbf{a}_{-i}^*), \ \forall i, \mathbf{s}$$

where $\mathbf{a} = (a_1, a_2, ..., a_N)$. Optimal actions $(\mathbf{a}^*)$ generate an equilibrium state transition

$$\begin{aligned} q(\mathbf{s}'|\mathbf{s}, \mathbf{a}^*) &= q[(s_1', s_2', ..., s_N')|(s_1, s_2, ..., s_N), (a_1^*, a_2^*, ..., a_N^*)] \\ &= \Pi_{i=1}^N p(s_i'|s_i, a_i^*) \end{aligned} \tag{2}$$

where the step from the first to the second equality derives from Assumption [2.1].

2.4 Symmetry and anonymity

The topic of symmetry and anonymity has been discussed in detail by Doraszelski and Satterthwaite (2007). Symmetry implies that the problem of a firm in state s_i when faced with an industry state $\mathbf{s}_{-i} = (s_1, .., s_{i-1}, s_{i+1}, s_N)$ is independent of the firm i itself and solely dependent on its state (firms are symmetric and all heterogeneity is captured through the state variables), i.e.

$$V_i(s_i, \mathbf{s}_{-i}) = V(s_i, \mathbf{s}_{-i}) \qquad \text{(Symmetry)}$$

Symmetry allows us to focus on the problem of solely one firm ($V(s_i, \mathbf{s}_{-i})$). Anonymity implies that firm i does not care about the identity of its competitors and its decision depends solely on the industry structure (distribution) it faces i.e.

$$\begin{aligned} V(s_1, ..., s_i, ...s_j &= x, ..., s_l = y, ...s_N) = \qquad \text{(Anonymity)} \\ V(s_1, ..., s_i, ...s_j &= y, ..., s_{jl} = x, ...s_N) \ \forall \ j, l \neq i. \end{aligned}$$

So for any two industry states with the same distribution the individual firm's problem is identical. Doraszelski and Satterthwaite (2007) provide sufficient conditions for this to hold in their Proposition 2, respectively that the period returns are symmetric and anonymous and that the transition function is also symmetric and anonymous.

3 Industry distribution

As explained in the previous section, the equilibrium for this game is a collection of strategies ($\mathbf{a}$) at each possible state ($\mathbf{s}$) which originate a Markov perfect equilibrium with transition $q(\mathbf{s}'|\mathbf{s}, \mathbf{a}^*)$. Solving the problem is computationally demanding since it requires the numerical solution of a dynamic programming problem in a potentially large state space $\mathfrak{s}^N$. The assumptions of anonymity and symmetry allow the problem to be rewritten in a different state space where

$$V(s_i, \mathbf{s}_{-i}) = V(s_i, f(\mathbf{s}_{-i}))$$

and

$$a^*(s_i, \mathbf{s}_{-i}) = a^*(s_i, f(\mathbf{s}_{-i}))$$

where $f(\mathbf{s}_{-i}) = (\frac{n_1}{N-1}, \frac{n_2}{N-1}, ..., \frac{n_K}{N-1})$, is a function mapping a particular industry state, $\mathbf{s}_{-i}$, into a probability space where $n_k = \sum_{j \neq i} \mathbf{1}(s_j = k)$ is the number of firms at each state $k = 1, ..., K$ and $f : \mathfrak{s}^{N-1} \rightarrow \mathfrak{F} \equiv \{0, \frac{1}{N-1}, ..., \frac{N-2}{N-1}, 1\}^{K-1}$. Notice that we are defining f over the vector $\mathbf{s}_{-i}$ and not solely over the scalar s. The purpose of this is to make explicit the relationship between a given state, $\mathbf{s}_{-i}$, and its distribution, f. We could equally define the vector $f_{-i}(s)$ analogously and make this a proper probability measure on the sample space $\{1, ..., K\}$ but in our case s is not a random variable.

The new state transition defined over the redefined state space is the probability density function $\hat{q} : \mathfrak{s} \times \mathfrak{F} \times \mathfrak{s} \times \mathfrak{F} \times \mathfrak{A}^N \rightarrow [0, 1]$ where a typical element $\hat{q}(s_i', f(\mathbf{s}_{-i}')|s_i, f(\mathbf{s}_{-i}), \mathbf{a})$ equals the probability that state $(s_i', f(\mathbf{s}_{-i}'))$ is reached from state $(s_i, f(\mathbf{s}_{-i}))$ when players choose actions $\mathbf{a}$. Equilibrium transition is $\hat{q}(s_i', f(\mathbf{s}_{-i}')|s_i, f(\mathbf{s}_{-\mathbf{i}}), \mathbf{a}^*)$. One of the disadvantages is that there is no general analytic expression for $\hat{q}$ even after specifying $p(s_i'|s_i, a_i^*)$ whereas in the previous case (using Assumption 2.1) $q(\mathbf{s}'|\mathbf{s}, \mathbf{a}^*) = \Pi_{i=1}^N p(s_i'|s_i, a_i^*)$. This means that we have to use numerical methods to calculate $\hat{q}$.

This redefinition has the particular advantage of reducing the dimension of the state space from the vector of states for each firm, $\mathbf{s} \in \mathfrak{s}^N$, with cardinality $(K-1)^N$ to the distribution of states, $(s_i, f(\mathbf{s}_{-i})) \in \mathfrak{s} \times \mathfrak{F}$, with cardinality

$$K * \binom{K+N-2}{N-1} \tag{3}$$

Example A.

Take a problem with $K = 10$ *states and* $N = 15$ *firms. The anonymity and symmetry assumptions reduce the dimension of the state space from* 9^{15} *to* $K * \binom{K+N-2}{N-1} \approx 8,200,000$ *which is a significant reduction. However, the cardinality of the industry state is still in the order of millions which is a computational challenge to solve* [2].

The magnitude of the problem grows with the dimension of the state space per firm (K) and the number of firms (N). I will solely focus on ways to circumvent the dimensionality problem which arises when the number of players grows. Notice that it does not mean the method only works for large N. In fact, as explained

[2] Matlab can store up to 15,000 by 15,000 single precision matrices in a dual-core 2.2 Ghz with 4GB RAM. Current computers are not able to solve even for this simple problem with 15 firms and one discrete state variable with dimension 10.

below, the quality of the approximation depends solely on the number of quantiles used. This is important because the method can be applied to industries with any number of firms.

Notice that even though using symmetry and anonymity severely reduces the cardinality of the problem, which no longer grows exponentially, it is also a fact that as seen by [3] when the number of firms increase, the state space continues to increase at a very high rate and the problem easily becomes unmanageable.

3.1 Quantiles

Define $F(\mathbf{s})$ as the cumulative distribution that completely describes $f(\mathbf{s}_{-i})$, where $f(\mathbf{s}_{-i}) = (\frac{n_1}{N-1}, \frac{n_2}{N-1}, ..., \frac{n_K}{N-1})$ then $F(\mathbf{s}_{-i}) = (\frac{n_1}{N-1}, \frac{n_1+n_2}{N-1}, ..., \frac{n_1+...+n_K}{N-1})$. Lets now define $P^j(\mathbf{s}_{-i})$ as the jth quantile for the industry distribution such that $\{\min P^j \in \mathfrak{s} : \frac{1}{N-1}\sum_{k=1}^{P^j} n_k \geq j,\ \forall j \in [0,1]\}$. The vector $P(\mathbf{s}_{-i}) = (P^{j1}(\mathbf{s}_{-i}), P^{j2}(\mathbf{s}_{-i}), ...P^{jR}(\mathbf{s}_{-i})) \in \mathfrak{P} \subset \mathfrak{s}^R$ is a vector with R quantiles for the industry state distribution at a given point in time. Notice that I am slightly abusing the usual definition by allowing the quantiles not to be equally spaced so that it fits the researcher's needs.

The quantile vector, $P : \mathfrak{s}^{N-1} \rightarrow \mathfrak{P} \subset \mathfrak{s}^R$ is a mapping from individual states into the R quantiles of the distribution of state $\mathbf{s}$ which characterizes how many firms there are in each state $k = 1, ..., K$. The cardinality of the set $\mathfrak{s} \times \mathfrak{P}$ is

$$K * \binom{K+R-1}{R} \tag{4}$$

We can now restate the original problem [1] using vector P to approximate the industry distribution

$$\hat{a}_i(s_i, P) = \arg\max_{a_i \in \mathfrak{A}} \pi(a_i, s_i, P) + \rho \hat{E}_P V(s_i', P'),\ \forall i, \mathbf{s} \tag{5}$$

where $\hat{E}_P$ is the expectation operator taken with respect to the industry distribution characterized by P.

$$\hat{E}_P V(s_i', P') = \sum_{k=1}^{K} \sum_{P' \in \mathfrak{P}} \hat{V}(s_i', P')\hat{q}(P'|P, s_i, a_i, \hat{\mathbf{a}}_{-i})p(s_i' = k|s_i, a_i)$$

$$\hat{q}[P'|P, s_i, a_i, \hat{\mathbf{a}}_{-i}] = \hat{q}[(P^{j1}(\mathbf{s}'_{-i}), ...P^{jR}(\mathbf{s}'_{-i}))|(P^{j1}(\mathbf{s}_{-i}), ...P^{jR}(\mathbf{s}_{-i})), s_i, (a_1, ..., a_N)]$$

and the value function is

$$\hat{V}_i(s_i, P) = \max_{a_i \in \mathfrak{A}} \pi(a_i, s_i, P) + \rho \hat{E}_P V(s'_i, P'),\ \forall i, \mathbf{s} \tag{6}$$

Theorem 3.1 *When $R + 1 = N$, and quantiles are equally spaced ($j_r = r/(R)$) the solution to problem [5] is identical to the solution for the original problem [1].*

Proof. If $R = N - 1$, the approximation matches the original industry distribution with r each category $j_r = r/(N-1)$ and so $P(\mathbf{s}_{-i}) \equiv \mathbf{s}_{-i}$, trivially making the solution to the quantile problem identical to the solution for the original problem. ■

Notice that the dimensionality of this problem is unrelated with the number of firms since the cardinality of set $\mathfrak{s} \times \mathfrak{P}$ solely depends on K and R. This result is important since the approximation will work for any number of firms, N, provided R is 'large enough' (below I discuss how to set R).

Using symmetry and anonymity the problem faced by an individual agent , i, becomes a function of the individual state (s_i) and the distribution for the industry structure ($f(\mathbf{s}_{-i})$) which is a variable in the set $[0, 1]^K$ with discrete support in $\mathfrak{s}^{N-1}$. The method discretizes this variable into R classes which resembles the discretization method used in numerically solving dynamic programming problems with continuous state variables.

Example A (contd).

Continuing the example with $K = 10$ states and $N = 15$ firms. If we use 5 quantiles (for example the percentiles $10, 30, 50, 70, 90$) the dimension of the state space is now $20,000$ which is significantly smaller than the previous 8 million. This illustrates the gains from discretizing the distribution in reducing the dimension of the problem, with the resulting computational advantages.

[**Insert figure 1**]

Figure 1 illustrates the example of an industry where $K = 5$ and $N = 11$. In this figure the cumulative distribution is represented for an industry in state $(N-1)f(\mathbf{s}_{-i}) = (1,1,0,3,5)$ so that the cumulative is $(N-1)F(\mathbf{s}_{-i}) = (1,2,2,5,10)$. The 25th, 50th and 75th percentiles are also drawn and they are $P = (P^{25}, P^{50}, P^{75}) = (4,4,5)$. Notice that since $N = 11$, if we use the $N-1$ equally spaced percentiles $\{P^{10}, P^{20}.., P^{90}, P^{100}\}$ the solution becomes exact.

3.2 A simulation based algorithm

Solving the dynamic game with quantiles as proposed above has to be done via simulations since contrary to the original problem where the matrix $q(\mathbf{s}'|\mathbf{s}, \mathbf{a}^*)$ can be solved analytically with $\Pi_{i=1}^{N} p(s_i'|s_i, a_i^*)$, now the matrix for $\hat{q}[P'|P, s_i, a_i, \hat{\mathbf{a}}_{-i}]$ cannot be solved analytically. I propose the following algorithm:

1. Start with a value for $\hat{q}^j[P'|P, s_i] \in \mathfrak{P} \times \mathfrak{P} \times \mathfrak{s}$ (e.g. this could be the identity matrix)

2. Solve the problem [6] and get $\hat{a}^j(s_i, P)$ and $\hat{V}^j(s_i, P)$ for all $(s_i, P) \in \mathfrak{s} \times \mathfrak{P}$

3. Set L industry configurations for $\mathbf{s}^l = (s_1^l, s_2^l, ..., s_N^l)$ and for each of them calculate $(s_i^l, P^l(\mathbf{s}_{-i}^l))$

4. Using $a^j(s, P)$ from step 2 and the primitive individual state transition take D draws from $p(s_i'|s_i, a_i)$

5. Construct a new vector for $\mathbf{s}'^{l,d} = (s_1'^{l,d}, s_2'^{l,d}, ..., s_N'^{l,d})$ and calculate $(P^{l,d}(\mathbf{s}_{-i}'^{l,d}))$ for each d and l.

6. Update the transition probability $\hat{q}^{j+1}[P'|P, s_i]$ and compare with $\hat{q}^j[P'|P, s_i]$ using some metric $||\hat{q}^{j+1} - \hat{q}^j||$

7. Stop if $||\hat{q}^{j+1} - \hat{q}^j|| < \varepsilon$ (where ε is some preset tolerance level) with solution $\hat{q}^* = \hat{q}^{j+1}$, otherwise update $\hat{q}^{j+1}$ and go back to 2.

Notice that one important condition is that in step 3, there are sufficient industry configurations in each class P^l so that all states in the set $\mathfrak{P}$ are updated correctly.

Convergence A convergence proof for this algorithm is not provided here. In the simulations, a linear combination in step 7 is normally sufficient to prevent the algorithm from diverging. In particular constructing $\hat{q}^{j+1} = \alpha\hat{q}^{j+1} + (1-\alpha)\hat{q}^{j}$ while setting a low value for $\alpha \in (0, 1]$. This is a method normally used in computational methods to prevent the algorithm from diverging (see for example Judd, 1998).

3.3 Optimal choice of R

There are mainly two sensible rules to use when setting the number of classes, R. First whenever N is small and can be solved with traditional methods, this should be prefered. Second, choose R to be as large as the computer can solve. Intuition tells us that the choice of R is problem specific since it will depend on K (if K is very large, R must be kept relatively small for computational reasons) and it will depend on the particular problem specification and parametrization. For example, if the expected long run distribution of the original problem is not 'well behaved', intuitively we would expect a crude approximation to perform poorly whereas if the long-run industry distribution is 'well behaved', few moments would be sufficient to approximate the original problem quite well. Neverthless depending on the research question under study, the quantiles should be chosen so that they capture the main features of industry structure. Furthermore, when $N \to \infty$, a small number of classes R might be sufficient to capture the industry distribution as Weintraub, Benkard and Van Roy (2008) have shown.

Notice also that the problem grows very quickly with R so in practice the number of classes should be kept relatively small. For example, Table 1 tabulates the cardinality of the problem for several combination of K and R and in particular if the state space is very large ($K = 50$) the maximum feasible number of classes one can have is $R = 2$ in order to be able to solve the problem in a common laptop computer. However, if the individual state space is reduced to $K = 15$ one can have $R = 5$ classes which already permits a very flexible approximation like $P = (P^{.05}, P^{.25}, P^{.50}, P^{.75}, P^{.95})$.

		R										
		1	2	3	4	5	6	7	8	9	10	11
K	1	1	1	1	1	1	1	1	1	1	1	1
	2	4	6	8	10	12	14	16	18	20	22	24
	3	9	18	30	45	63	84	108	135	165	198	234
	4	16	40	80	140	224	336	480	660	880	1,144	1,456
	5	25	75	175	350	630	1,050	1,650	2,475	3,575	5,005	6,825
	6	36	126	336	756	1,512	2,772	4,752	7,722	12,012	18,018	26,208
	7	49	196	588	1,470	3,234	6,468	12,012	21,021	35,035	56,056	86,632
	8	64	288	960	2,640	6,336	13,728	27,456	51,480	91,520	$1.6\text{x}10^{05}$	$2.5\text{x}10^{05}$
	9	81	405	1,485	4,455	11,583	27,027	57,915	$1.2\text{x}10^{05}$	$2.2\text{x}10^{05}$	$3.9\text{x}10^{05}$	$6.8\text{x}10^{05}$
	10	100	550	2,200	7,150	20,020	50,050	$1.1\text{x}10^{05}$	$2.4\text{x}10^{05}$	$4.9\text{x}10^{05}$	$9.2\text{x}10^{05}$	$1.7\text{x}10^{06}$
	15	225	1,800	10,200	45,900	174,420	$5.8\text{x}10^{05}$	$1.7\text{x}10^{06}$	$4.8\text{x}10^{06}$	$1.2\text{x}10^{07}$	$2.9\text{x}10^{07}$	$6.7\text{x}10^{07}$
	20	400	4,200	30,800	177,100	$8.5\text{x}10^{05}$	$3.5\text{x}10^{06}$	$1.3\text{x}10^{07}$	$4.4\text{x}10^{07}$	$1.4\text{x}10^{08}$	$4.0\text{x}10^{08}$	$1.1\text{x}10^{09}$
	25	625	8,125	73,125	$5.1\text{x}10^{05}$	$3.0\text{x}10^{06}$	$1.5\text{x}10^{07}$	$6.6\text{x}10^{07}$	$2.6\text{x}10^{08}$	$9.6\text{x}10^{08}$	$3.3\text{x}10^{09}$	$1.0\text{x}10^{10}$
	30	900	13,950	$1.5\text{x}10^{05}$	$1.2\text{x}10^{06}$	$8.3\text{x}10^{06}$	$4.9\text{x}10^{07}$	$2.5\text{x}10^{08}$	$1.2\text{x}10^{09}$	$4.9\text{x}10^{09}$	$1.9\text{x}10^{10}$	$6.9\text{x}10^{10}$
	50	2,500	$6.4\text{x}10^{04}$	$1.1\text{x}10^{06}$	$1.5\text{x}10^{07}$	$1.6\text{x}10^{08}$	$1.4\text{x}10^{09}$	$1.2\text{x}10^{10}$	$8.3\text{x}10^{10}$	$5.3\text{x}10^{11}$	$3.1\text{x}10^{12}$	$1.7\text{x}10^{13}$
	100	10,000	$5.1\text{x}10^{05}$	$1.7\text{x}10^{07}$	$4.4\text{x}10^{08}$	$9.2\text{x}10^{09}$	$1.6\text{x}10^{11}$	$2.4\text{x}10^{12}$	$3.3\text{x}10^{13}$	$3.9\text{x}10^{14}$	$4.3\text{x}10^{15}$	$4.3\text{x}10^{16}$
	150	22,500	$1.7\text{x}10^{06}$	$8.6\text{x}10^{07}$	$3.3\text{x}10^{09}$	$1.0\text{x}10^{11}$	$2.6\text{x}10^{12}$	$5.8\text{x}10^{13}$	$1.1\text{x}10^{15}$	$2.0\text{x}10^{16}$	$3.2\text{x}10^{17}$	$4.7\text{x}10^{18}$

Notes: The area where combinations of R and K are smaller than 15,000 is the maximum matrix size Matlab can handle in a desktop with 4GB RAM.

Table 1: Different combinations of state size (K) and number of quantiles (R) and respective dimensionality of the industry state space.

3.4 Advantages

Several advantages result from using the 'quantile approximation'. Besides the obvious computational advantage that habilitates one to solve problems which were not possible before, there is also the important advantage that this can be done flexibly capturing different industry structures. Imagine for example that the researcher is interested in the problem of market leadership and concentration. A reasonable specification would include the top quantile of the distribution (e.g. P^{99}) which would capture the existence of these market leaders. Therefore, the approximation can be constructed in a flexible way by the researcher such that it captures the main features of the problem under study. This is a important advantage when compared with other existing methods.

4 Simulation Results

In this section the performance of the proposed methodology is compared with the full solution method. The model is very similar to Ericson and Pakes (1995). In particular the primitives for the demand system, cost function and state transition will be specified. There is only one state variable per firm, $s_i \in [1, 2, ..., K]$, which denotes product characteristic. Firms can then invest an amount, x, in trying to improve the quality of their product.

4.1 Demand

Products are differentiated and there are m consumers in the market. Consumer j receives utility u_{ij} from consuming the good produced by firm i

$$u_{ij} = \beta s_i - \alpha p_i + v_{ij}$$

where v_{ij} are extreme value independent and identically distributed preference shocks. Integrating over v_{ij}, market shares can be written as

$$\mu_i = \frac{\exp(\beta s_i - \alpha p_i)}{1 + \sum_{j=1}^{N} \exp(\beta s_j - \alpha p_j)}$$

Marginal costs of production, c, are constant and equal for all firms. The Nash equilibrium for the pricing game satisfies the following equation:

$$\max_{p_i^*} \frac{\exp(\beta s_i - \alpha p_i)}{1 + \sum_{j=1}^{N} \exp(\beta s_j - \alpha p_j)} m(p_i^* - c)$$

and equilibrium prices are

$$p_i^* = c + \frac{1}{\alpha(1 - \mu_i)}$$

4.2 Transition function

Firms can invest x_i to improve the quality of their product at a cost ax_i. The quality of their good will go up by one unit to $s_i + 1$ with probability $\frac{hx}{1+hx}$ and will stay the same with probability $\frac{1}{1+hx}$. At the same time there is an exogenous probability δ that the quality of the good goes down. The individual state transition $p(s_i'|s_i, a_i)$ is

$$p(s_i' = y | s_i = s, x) = \begin{cases} \frac{(1-\delta)hx}{1+x} & \text{if } y = s+1 \\ \frac{(1-\delta)+\delta hx}{1+x} & \text{if } y = s \\ \frac{\delta}{1+hx} & \text{if } y = s-1 \end{cases}$$

4.3 Parametrization

The parameters used in the simulations are the following

N	10
K	5
β	2
α	10
m	10
δ	0.7
h	3
a	0.02
ρ	0.9

The cardinality of the original state space is $4^{10} \approx 1$ million and for the model imposing symmetry and anonymity this is $3,575$. Finally for the model using the approximation it is 630 for five percentiles ($R = 5$) and 175 for three percentiles ($R = 3$). The reduction in the state size is extremely significant.

4.4 Results

In this section I compare the results for 3 different specifications. First, I solve the original EP framework as specified in [1]. Second, I solve the model approximating the industry distribution with five percentiles $[P^{10}, P^{30}, P^{50}, P^{70}, P^{90}]$ and finally I solve a third model using only three percentiles $[P^{25}, P^{50}, P^{75}]$.

The results, presented in figures 2 to 7, show that both the value and the investment functions are very well approximated using the distribution moments. Notice that these figures are one dimensional representations constructed by stacking the whole multidimensional industry state along a single dimension and for this reason both value functions, policy functions and industry distributions are not monotonic since the natural order (and monotonicity) is absent in multidimensional Euclidean spaces. Basically each (multidimensional) state has been redefined over a single dimension. To get the monotonic representation I have sorted the vector of values (and investment) along the dimension for the original model. The long run industry distribution is also very well approximated. Figures 2 to 4 present the value function, investment function and long run distribution respectively for the 5 percentile case and figures 5 to 7 present the same functions for the 3 percentile case. These results confirm that using distribution moments can effectively break the 'curse' of dimensionality while providing a very good approximation to the original problem. As expected, there is an improvement going from three to five percentiles.

[**insert figures 2-7**]

If we define $g(\mathbf{s}_k) = \Pr[\mathbf{s} = \mathbf{s}_k]$ to be the long run industry distribution as the distribution generated by the equilibrium industry transition $q(\mathbf{s}'|\mathbf{s}, \mathbf{a}^*)$, the expected long run industry state, $f^{LR}(\mathbf{s}) = \sum_k g(\mathbf{s}_k) f(\mathbf{s}_k)$, for this parametrization is

$$f^{LR}(\mathbf{s}) = [\ 0.32 \quad 0.06 \quad 0.08 \quad 0.17 \quad 0.37\ ]$$

which means that the industry is expected to be in a ('asymetric') state with most firms either with low (1) or high (5) quality. There is a tendency for a 'double peaked' equilibrium industry distribution, i.e., some firms very high quality and some firms with very low quality. While our intuition tells us that a good approximation could be produced for a 'single peaked' (well behaved) equilibrium industry distribution, it is very satisfying that this is true even for 'non-well behaved' industry structures.

Throughout this section I have simplified the model not allowing entry and exit and using one step increase/decrease for the individual state transition functions. This was only for expositional purposes and the results from section [3.1] extend to models with entry and exit and general individual state transitions.

5 Extensions

5.1 Private information

Information has been assumed to be common to all players. However, in some cases information might be private. The extension for privately observed independent and identically distributed signals is trivial. Nevertheless, the extension to non-transitory private information brings non-trivial added complexity due to the fact that it becomes a learning game. In general a model with learning will lead to an history dependent (non-Markovian) equilibrium. Santos (2008) develops a model with private information where players only observe the aggregate state. Under certain conditions this leads to a Markovian evolution for the aggregate state which is a sufficient statistic to characterize industry equilibrium evolution.

5.2 Multiple state variables

When there are multiple state variables per firm the above method is not directly applicable. The quantile function is not defined in the multidimensional case because the cumulative distribution is not invertible (for a review of methods to model multivariate quantile functions, see Serfling, 2002). Neverthless, a similar approach can be used and several options are available: First the most straightforward way is to reshape the multiple dimensions along a single dimension and use the same methodology as for the unidimensional case. The main difference is that the natural order between the states is now meaningless since this order relation in the new (unidimensional) state space has been modified. This is because the natural order is absent for Euclidean spaces with dimensions greater than one. For example, imagine a case with two state variables (s^1, s^2) that can take K^1 and K^2 different values. We can construct a third state variable s^3 which can take $K^3 = K^1 \times K^2$ different values by simply stacking the two states together. Whereas s^1 and s^2 had a meaningful natural order in the set $(K^i, \leq)$, this new state variable has no meaningful order relation in its set. What this means is that the cumulative can still represent the industry distribution but looses its traditional interpretation making the notion of what is a 'good' choice of quantiles less clear. A second option is to construct the quantile function for the marginal distributions in the same way it was done for the unidimensional case and use the Copula function to join the two distributions. Finally, a third option is to discretize the probability function into probability classes (e.g. 0%-15%, 15%-45%, 45%-100%) instead of discretizing the cumulative density.

5.3 Continuous state variables

I have focused here in discrete state games because using numerical solutions normally requires the discretization of the state space. This makes the discrete state the natural framework to use besides being simpler. Notice that one alternative to the quantiles approach would be the use of distribution moments. From a practical perspective these two approaches would be similar but from a theoretical perspective this is a harder problem to address since the state variables are now continuous but the number of firms at each period is finite unless we extend to infinitesimal firms.

6 Conclusion

In this paper I have proposed a method to circumvent the increase in the cardinality of the state space in infinite horizon discrete state dynamic games. I have restricted to symmetric and anonymous Markovian games. Most games in this class cannot be solved analytically and the dimensionality of the problem grows with the number of players, making them also computationally difficult to solve. It is known that for these games, under symmetry and anonymity, the state space can be (re)defined as the industry state distribution (Doraszelski and Satterthwaite, 2007). I have proposed using quantiles of this distribution as an approximation/discretization method to the original framework. The main advantage of this is that the cardinality of the problem becomes completely unrelated to the number of firms. I have also shown that the approximation becomes exact as the number of quantiles grows. Finally, I have illustrated with a simple example that this approximation works reasonably well even with very few quantiles. I hope this result opens the venue for the application (and estimation) of dynamic games to a set of situations where this was previously impossible.

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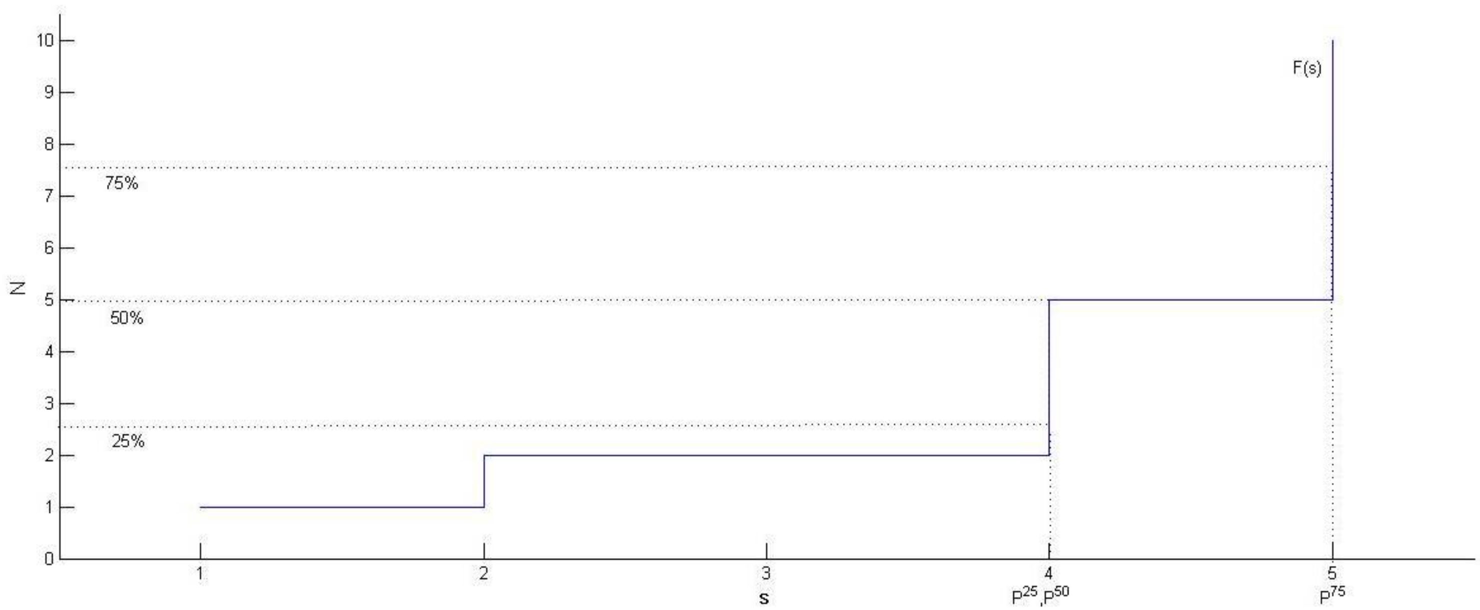

Figure 1 – Industry Distribution and Percentile approximation: The cumulative function for the industry state, F(s), is represented for one case with K=5, F(s)=[1,2,2,5,10]. The 25th, 50th and 75th percentiles are also drawn [P(25) P(50) P(75)]=[4,4,5]

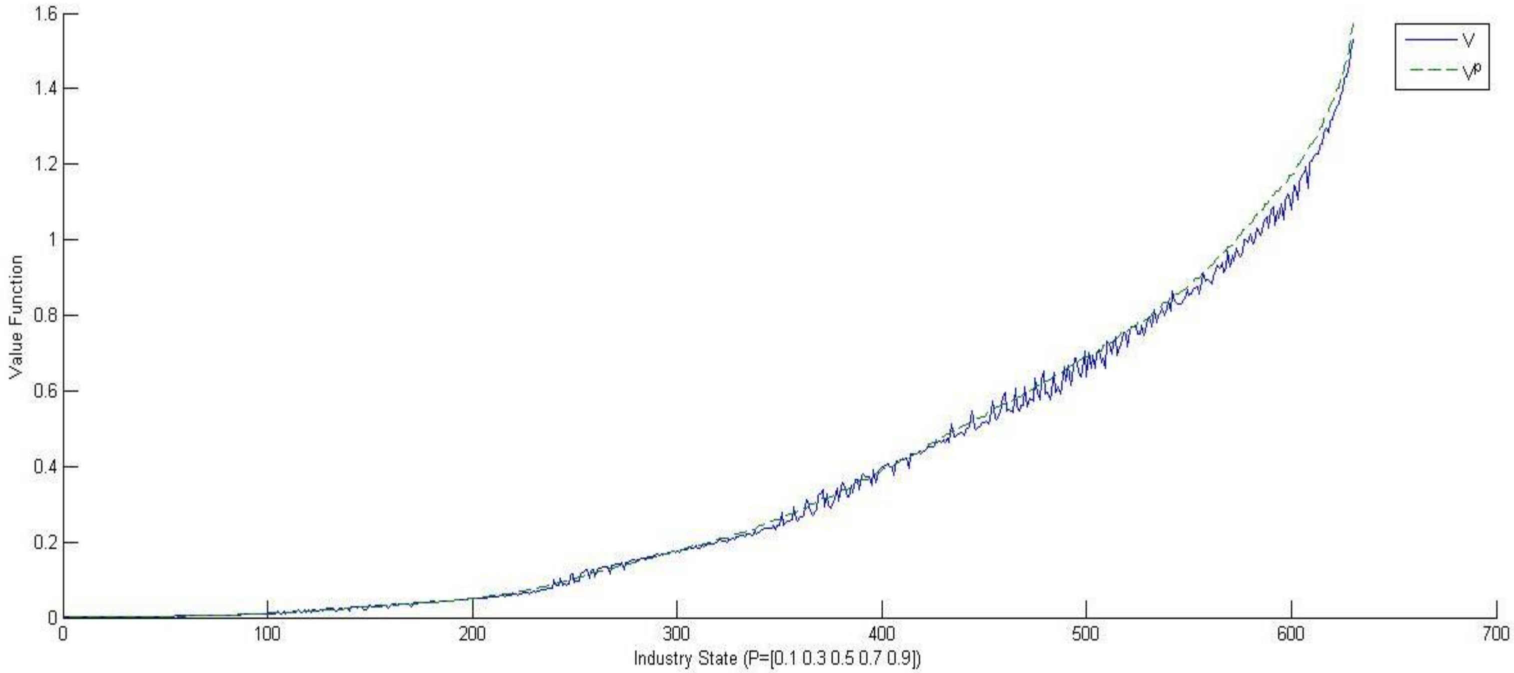

Figure 2 – Value Function for the original problem (solid line) and using the percentile approximation (dashed line) over the state space spanned by the five percentiles

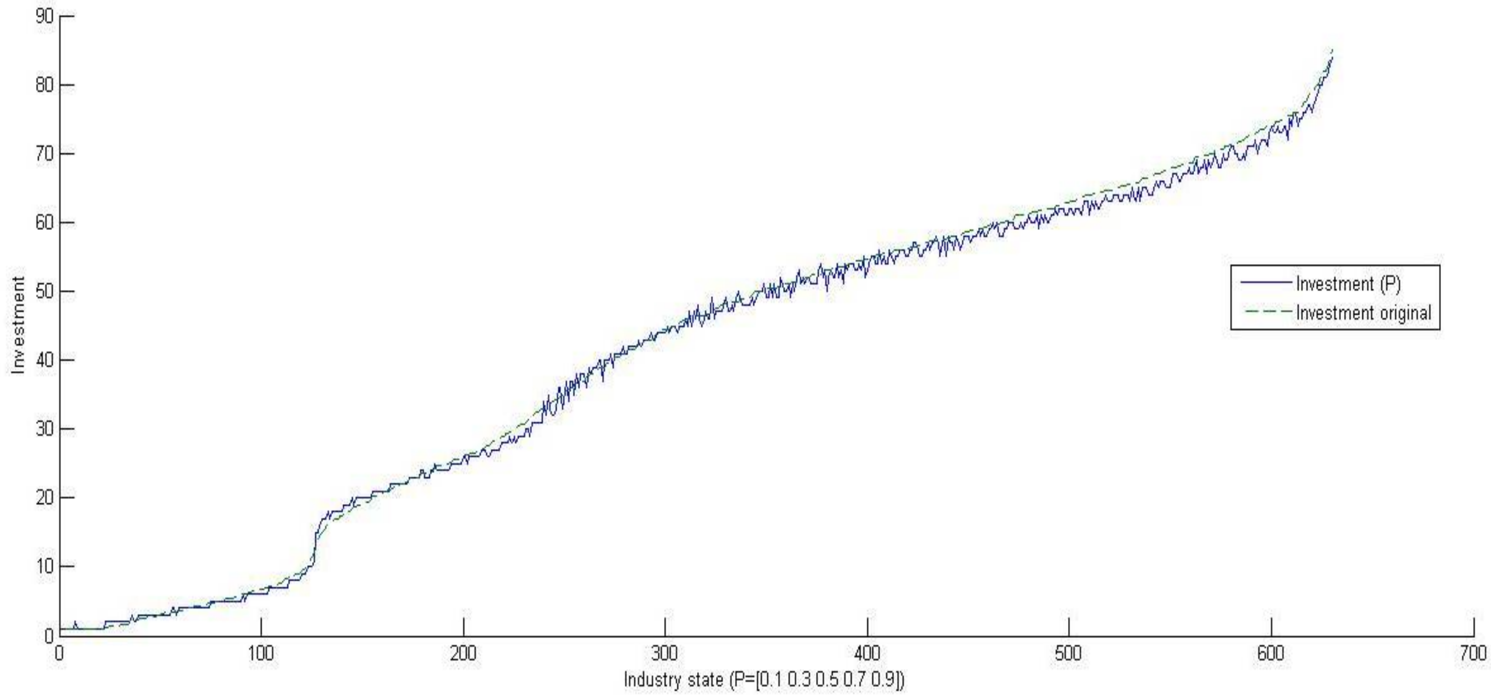

Figure 3 – Investment Function for the original problem (solid line) and using the percentile approximation (dashed line) over the state space spanned by the five percentiles

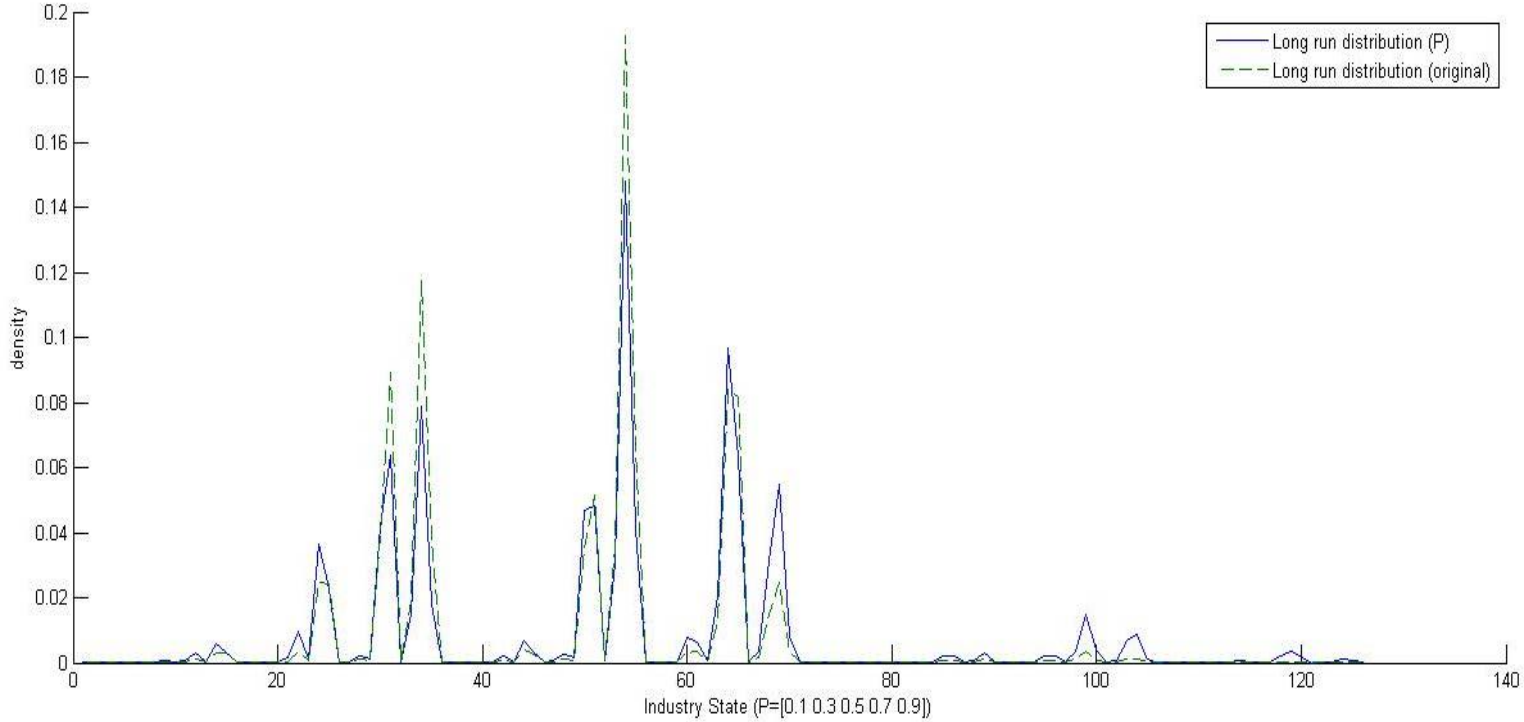

Figure 4 – Long run industry distribution generated for the original problem (dashed line) and using the percentile approximation (solid line) over the state space spanned by the five percentiles

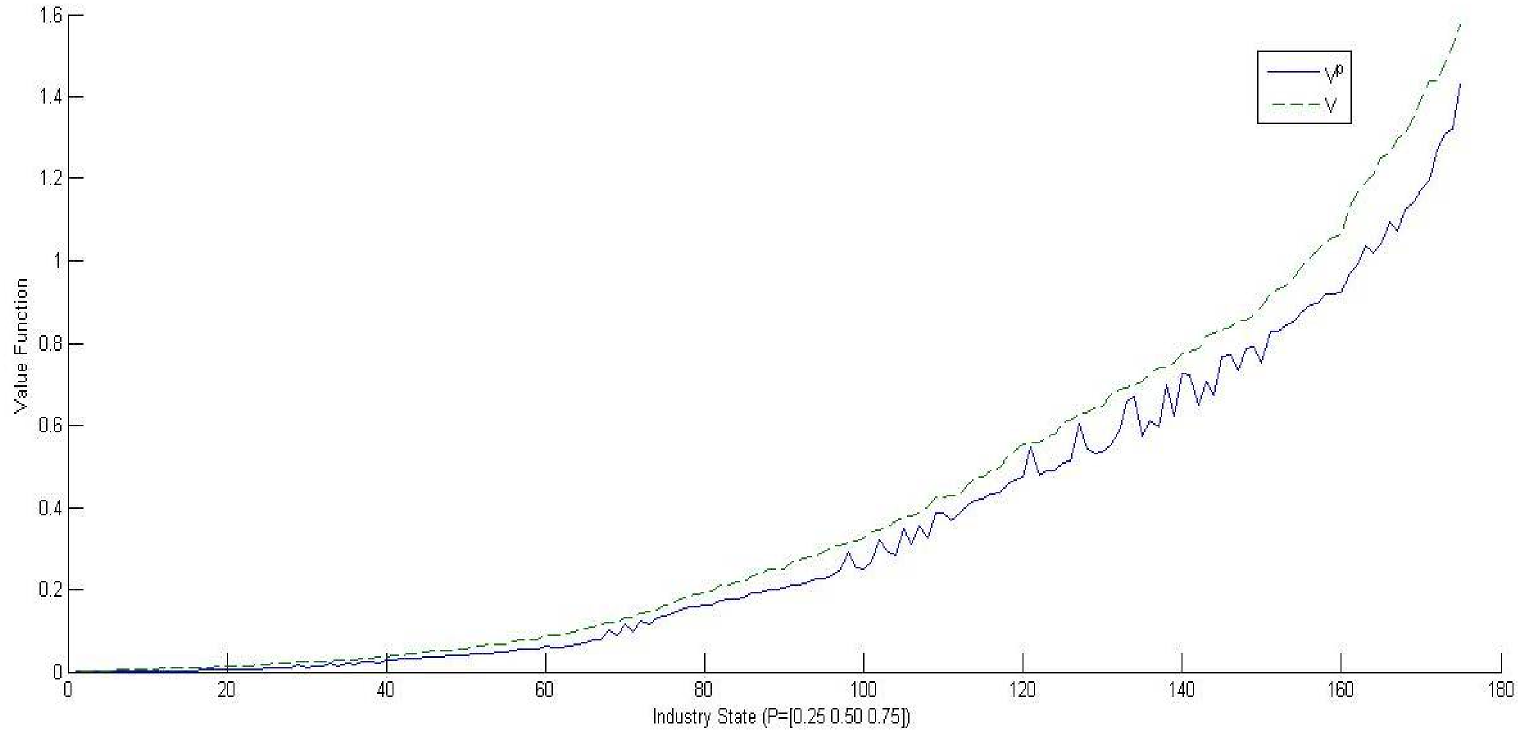

Figure 5 – Value Function for the original problem (dashed line) and using the percentile approximation (solid line) over the state space spanned by the three percentiles

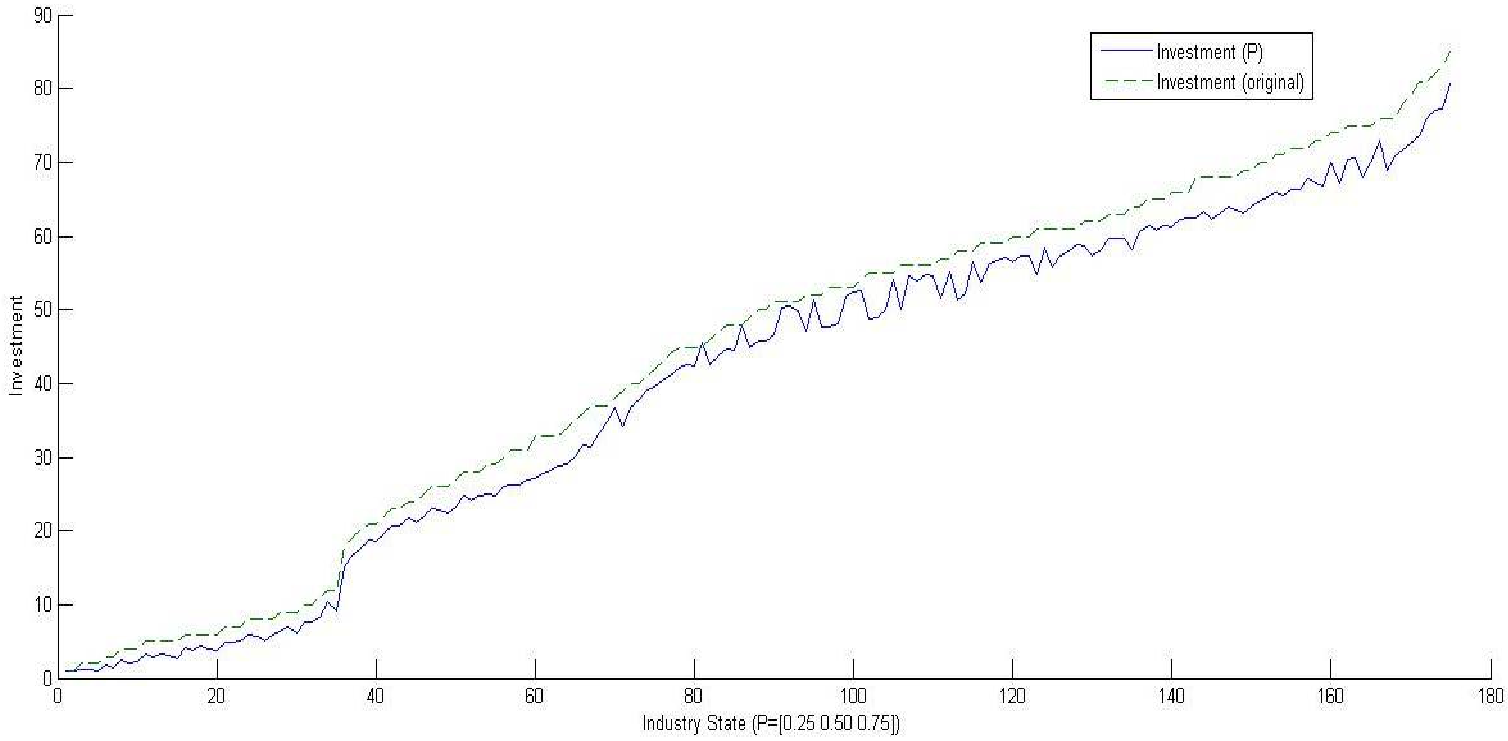

Figure 6 – Investment Function for the original problem (dashed line) and using the percentile approximation (solid line) over the state space spanned by the three percentiles

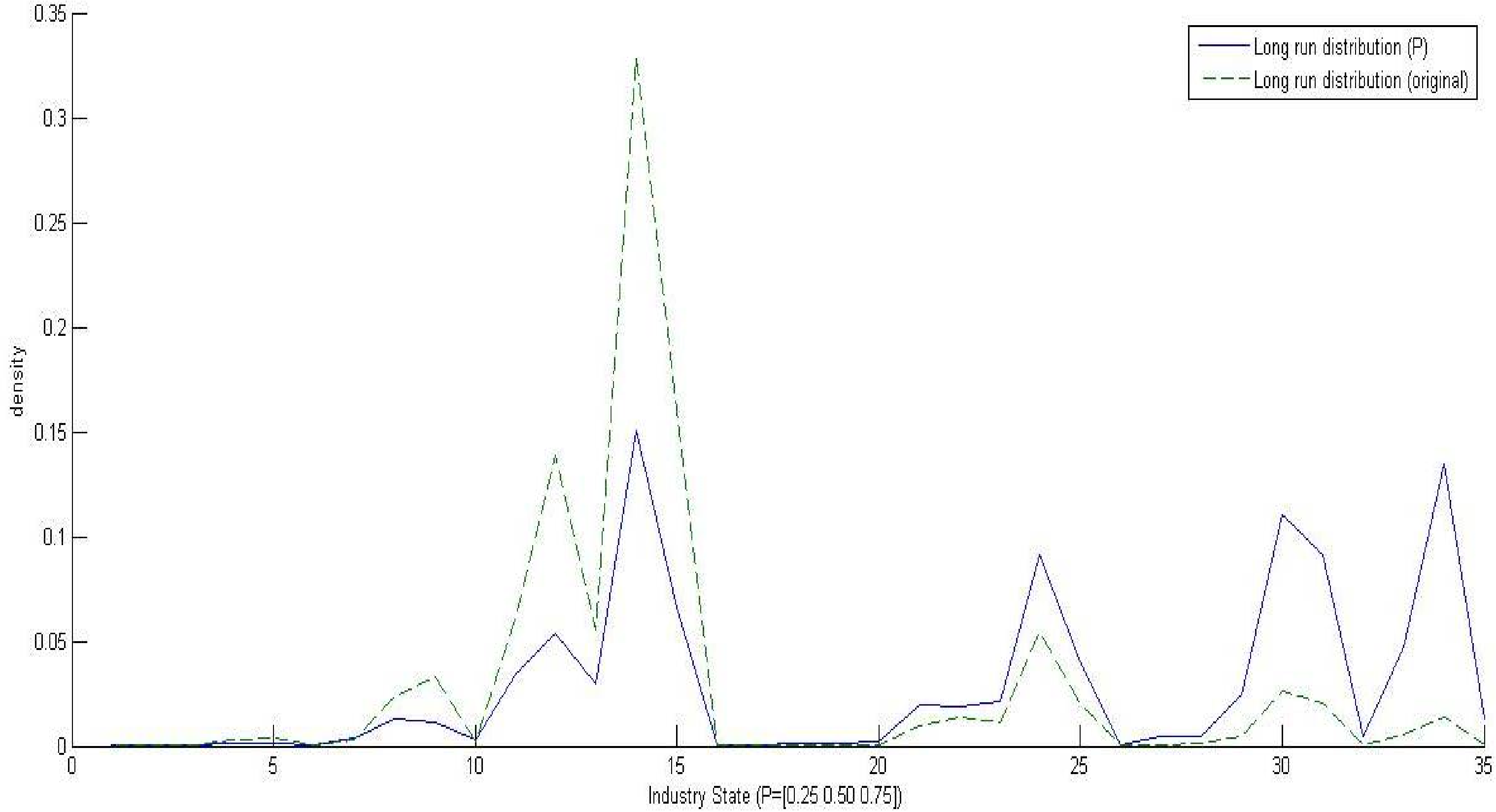

Figure 7 – Long run industry distribution generated for the original problem (dashed line) and using the percentile approximation (solid line) over the state space spanned by the three percentiles